QCD and Collider Physics
Lecture I: Monte Carlo programs and Jets

Troisième Cycle de la Physique en Suisse Romande
November-December, 2005

Keith Ellis

`ellis@fnal.gov`

Fermilab/CERN

Slides available from   http://theory.fnal.gov/people/ellis/3Cycle/

Bibliography

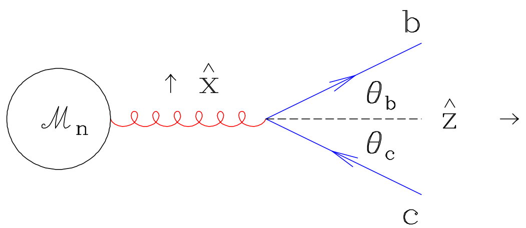
QCD and Collider Physics
(Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology)

by R. K. Ellis, W.J. Stirling and B.R. Webber

Monte Carlo and Jets

- Parton Branching
- DGLAP equation
  - ⋆ Quarks and gluons
  - ⋆ Solution by moments
- Sudakov form factor
  - ⋆ Infrared cutoff
- Monte Carlo method
- Soft gluon emission
  - ⋆ Angular ordering
  - ⋆ Coherent branching

Parton branching - kinematics



$$
\begin{aligned}
p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\
p_b &= (E_b, +E_b \sin\theta_b, 0, +E_b \cos\theta_b) \\
p_c &= (E_c, -E_c \sin\theta_c, 0, +E_c \cos\theta_c)
\end{aligned}
$$

- the kinematics and notation for the branching of parton a into $b + c$. We assume that

$$p_b^2,\, p_c^2 \ll p_a^2 \equiv t$$

- a is an outgoing parton, which is called timelike branching since $t > 0$.
- The opening angle is $\theta = \theta_b + \theta_c$. Defining the energy fraction as

$$z = E_b/E_a = 1 - E_c/E_a \ ,$$

we have for small angles, $t = 2E_bE_c(1 - \cos\theta) = z(1-z)E_a^2\theta^2$

- using transverse momentum conservation,

$$\theta = \frac{1}{E_a}\sqrt{\frac{t}{z(1-z)}} = \frac{\theta_b}{1-z} = \frac{\theta_c}{z} \ .$$

Dirac eqn. Massless fermions

- The fermions involved in high energy processes can often be taken to be massless.
- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

  The Weyl representation is more suitable at high energy

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

  In the Weyl representation upper and lower components have different helicities.
- Both representations satisfy the same commutation relations.

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

- in the Weyl representation $\gamma^0 \gamma^i = \begin{pmatrix} \sigma^i & \mathbf{0} \\ \mathbf{0} & -\sigma^i \end{pmatrix}$. σ are the Pauli matrices.

- The massless spinors solns of Dirac eqn are

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-}e^{i\varphi_p} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-}e^{-i\varphi_p} \\ -\sqrt{p^+} \end{bmatrix},$$

where

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+p^-}}, \qquad p^\pm = p^0 \pm p^3.$$

In this representation the Dirac conjugate spinors are

$$\overline{u_+}(p) \equiv u_+^\dagger(p)\gamma^0 = \left[0, 0, \sqrt{p^+}, \sqrt{p^-}e^{-i\varphi_p}\right]$$

$$\overline{u_-}(p) = \left[\sqrt{p^-}e^{i\varphi_p}, -\sqrt{p^+}, 0, 0\right]$$

- Normalization

$$u_\pm^\dagger u_\pm = 2p^0$$

Branching probabilities

- Consider the case where

$$
\begin{aligned}
p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\
p_b &\sim (E_b, +E_b\theta_b, 0, +E_b) \\
p_c &\sim (E_c, -E_c\theta_c, 0, +E_c)
\end{aligned}
$$

Thus for example

$$u_+^\dagger(p) = \sqrt{2E_b}\left[1, \frac{\theta_b}{2}, 0, 0\right]$$

and

$$u_+(p_c) \equiv v_-(p_c) = \sqrt{2E_c}\begin{bmatrix} 1 \\ -\frac{\theta_c}{2} \\ 0 \\ 0 \end{bmatrix}$$

Hence for polarization vectors $\varepsilon_{in} = (0, 1, 0, 0), \varepsilon_{out} = (0, 0, 1, 0)$

$$g\bar{u}_+^b\ \gamma^0\gamma^1\ v_-^c = g\sqrt{4E_bE_c}\left(1, \frac{\theta_b}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{\theta_c}{2} \end{pmatrix} = -g\sqrt{E_bE_c}(\theta_b - \theta_c)$$

$$-g\bar{u}_{+}^{b}\gamma_\mu \varepsilon_a^{p\text{in}^\mu} v_-^c = g\sqrt{E_bE_c}(\theta_b-\theta_c) = g\sqrt{z(1-z)}(1-2z)E_a\theta\ ,$$

$$-g\bar{u}_{+}^{b}\gamma_\mu \varepsilon_a^{p\text{out}^\mu} v_-^c = ig\sqrt{E_bE_c}(\theta_b+\theta_c) = ig\sqrt{z(1-z)}E_a\theta\ ,$$

and the matrix element relation for the branching is

$$|\mathcal{M}_{n+1}|^2 \sim \frac{g^2}{t} T_R F(z;\varepsilon_a,\lambda_b,\lambda_c)|\mathcal{M}_n|^2$$

where the colour factor is now Tr$(t^At^A)/8 = T_R = 1/2$. The non-vanishing functions $F(z;\varepsilon_a,\lambda_b,\lambda_c)$ for quark and antiquark helicities λ_b and λ_c are

ε_a	λ_b	λ_c	$F(z;\varepsilon_a,\lambda_b,\lambda_c)$
in	$\pm$	$\mp$	$(1-2z)^2$
out	$\pm$	$\mp$	1

Summing over the polarizations we get

$$2\Big[(1-2z)^2+1\Big] = 4(z^2+(1-z)^2).$$

Branching probabilities

$$\int \frac{d\phi}{2\pi} CF = \hat{P}_{ba}(z)$$

where $\hat{P}_{ba}(z)$ is the appropriate splitting function

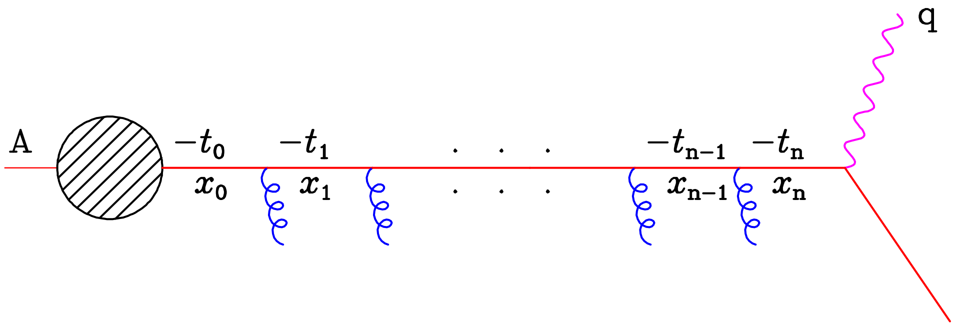
$$d\sigma_{n+1} = d\sigma_n \, \frac{dt}{t} \, dz \, \frac{\alpha_S}{2\pi} \, \hat{P}_{ba}(z) \ .$$

- Including all the color factors we find the results for the unregulated branching probabilities.

$$\begin{aligned}
\hat{P}_{qq}(z) &= C_F \left[\frac{1+z^2}{(1-z)}\right] \ , \\
\hat{P}_{qg}(z) &= T_R \left[z^2 + (1-z)^2\right] \ , \quad T_R = \frac{1}{2} \ , \\
\hat{P}_{gq}(z) &= C_F \left[\frac{1+(1-z)^2}{z}\right] \ , \\
\hat{P}_{gg}(z) &= C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z\,(1-z)\right]
\end{aligned}$$

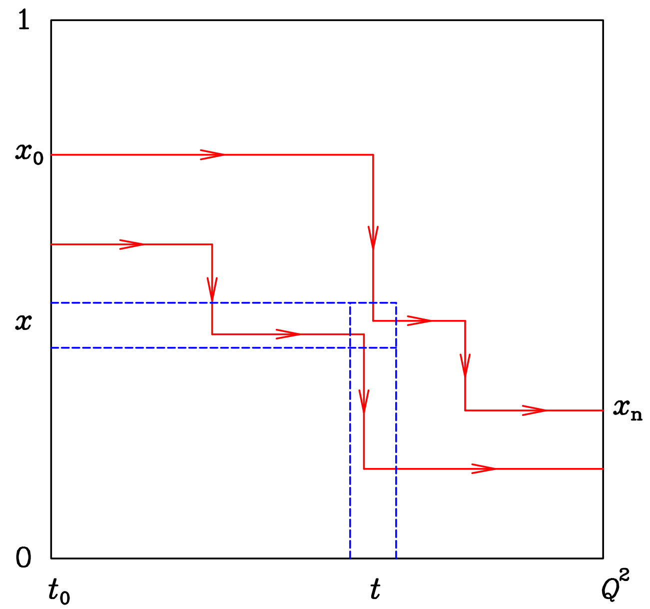
DGLAP equation

- Consider enhancement of higher-order contributions due to multiple small-angle parton emission, for example in deep inelastic scattering (DIS)



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.
- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful later for Monte Carlo simulation.



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.

Change in parton distribution

- Consider change in the parton distribution $D(x,t)$ when t is increased to $t+\delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.
- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned}\delta D_{\text{in}}(x,t) &= \frac{\delta t}{t}\int_x^1 dx'\,dz\frac{\alpha_S}{2\pi}\hat{P}(z)D(x',t)\,\delta(x-zx') \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t)\end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned}\delta D_{\text{out}}(x,t) &= \frac{\delta t}{t}D(x,t)\int_0^x dx'\,dz\frac{\alpha_S}{2\pi}\hat{P}(z)\,\delta(x'-zx) \\ &= \frac{\delta t}{t}D(x,t)\int_0^1 dz\frac{\alpha_S}{2\pi}\hat{P}(z)\end{aligned}$$

Change in parton distribution

- Change in population of element is

$$\begin{aligned}\delta D(x,t) &= \delta D_{\text{in}} - \delta D_{\text{out}} \\ &= \frac{\delta t}{t}\int_0^1 dz\, \frac{\alpha_S}{2\pi}\hat{P}(z)\left[\frac{1}{z}D(x/z,t) - D(x,t)\right] .\end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dx\; f(x)\, g(x)_+ \;=\; \int_0^1 dx\, [f(x) - f(1)]\, g(x) \;.$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+ \;,$$

DGLAP

We obtain the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)D(x/z,t)\ .$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t. In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

Quarks and gluons

- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P_{ij}(z)D_j(x/z,t)\ .$$

- Quark ($i = q$) can enter element via either $q \to qg$ or $g \to q\bar{q}$, but can only leave via $q \to qg$. Thus plus-prescription applies only to $q \to qg$ part, giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2+(1-z)^2\right] \end{aligned}$$

- Gluon can arrive either from $g \to gg$ (2 contributions) or from $q \to qg$ (or $\bar{q} \to \bar{q}g$). Thus number arriving is

$$\delta D_{g,\mathrm{in}} = \frac{\delta t}{t}\int_0^1 dz \frac{\alpha_S}{2\pi}\left\{\hat{P}_{gg}(z)\left[\frac{D_g(x/z,t)}{z} + \frac{D_g(x/(1-z),t)}{1-z}\right] + \frac{\hat{P}_{qq}(z)}{1-z}\left[D_q\left(\frac{x}{1-z},t\right) + D_{\bar{q}}\left(\frac{x}{1-z},t\right)\right]\right\}$$

$$= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\left\{2\hat{P}_{gg}(z)D_g\left(\frac{x}{z},t\right) + \hat{P}_{qq}(1-z)\left[D_q\left(\frac{x}{z},t\right) + D_{\bar{q}}\left(\frac{x}{z},t\right)\right]\right\},$$

- Gluon can leave by splitting into either gg or $q\bar{q}$, so that

$$\delta D_{g,\mathrm{out}} = \frac{\delta t}{t} D_g(x,t)\int_0^1 dz \frac{\alpha_S}{2\pi}\left[\hat{P}_{gg}(z) + N_f \hat{P}_{qg}(z)\, dz\right] .$$

- After some manipulation we find

$$P_{gg}(z) = 2C_A\left[\left(\frac{z}{1-z} + \frac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z} + \frac{1}{2}z(1-z)\right] - \frac{2}{3}N_f T_R\,\delta(1-z) ,$$

$$P_{gq}(z) = P_{g\bar{q}}(z) = \hat{P}_{qq}(1-z) = C_F\frac{1+(1-z)^2}{z} .$$

- Using definition of the plus-prescription, can check that

$$
\begin{aligned}
\left(\frac{z}{1-z}+\frac{1}{2}z(1-z)\right)_+ &= \frac{z}{(1-z)_+}+\frac{1}{2}z(1-z)+\frac{11}{12}\delta(1-z)\\
\left(\frac{1+z^2}{1-z}\right)_+ &= \frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\ ,
\end{aligned}
$$

so P_{qq} and P_{gg} can be written in more common forms

$$
\begin{aligned}
P_{qq}(z) &= C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right]\\
P_{gg}(z) &= 2C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+\frac{1}{6}(11C_A-4N_fT_R)\,\delta(1-z)\ .
\end{aligned}
$$

Solution by moments

- Given $D_i(x,t)$ at some scale $t = t_0$, factorized structure of DGLAP equation means we can compute its form at any other scale.
- One strategy for doing this is to take moments (Mellin transforms) with respect to x:

$$\tilde{D}_i(N,t) = \int_0^1 dx\; x^{N-1}\; D_i(x,t)\;.$$

  Inverse Mellin transform is

$$D_i(x,t) = \frac{1}{2\pi i}\int_C dN\; x^{-N}\;\tilde{D}_i(N,t)\;,$$

  where contour C is parallel to imaginary axis to right of all singularities of integrand.
- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t\frac{\partial}{\partial t}\tilde{D}_i(x,t) = \sum_j \gamma_{ij}(N,\alpha_S)\tilde{D}_j(N,t)$$

Anomalous dimensions

- The moments of splitting functions give PT expansion of anomalous dimensions γ_{ij}:

$$\begin{aligned}
\gamma_{ij}(N,\alpha_S) &= \sum_{n=0}^{\infty} \gamma_{ij}^{(n)}(N) \left(\frac{\alpha_S}{2\pi}\right)^{n+1} \\
\gamma_{ij}^{(0)}(N) &= \tilde{P}_{ij}(N) = \int_0^1 dz\; z^{N-1}\; P_{ij}(z)
\end{aligned}$$

- From above expressions for $P_{ij}(z)$ we find

$$\begin{aligned}
\gamma_{qq}^{(0)}(N) &= C_F\left[-\frac{1}{2} + \frac{1}{N(N+1)} - 2\sum_{k=2}^{N}\frac{1}{k}\right] \\
\gamma_{qg}^{(0)}(N) &= T_R\left[\frac{(2+N+N^2)}{N(N+1)(N+2)}\right] \\
\gamma_{gq}^{(0)}(N) &= C_F\left[\frac{(2+N+N^2)}{N(N^2-1)}\right] \\
\gamma_{gg}^{(0)}(N) &= 2C_A\left[-\frac{1}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - \sum_{k=2}^{N}\frac{1}{k}\right] - \frac{2}{3}N_f T_R\,.
\end{aligned}$$

Scaling violation

- Consider combination of parton distributions which is flavour non-singlet, e.g. $D_V = D_{q_i} - D_{\bar{q}_i}$ or $D_{q_i} - D_{q_j}$. Then mixing with the flavour-singlet gluons drops out and solution for fixed α_S is

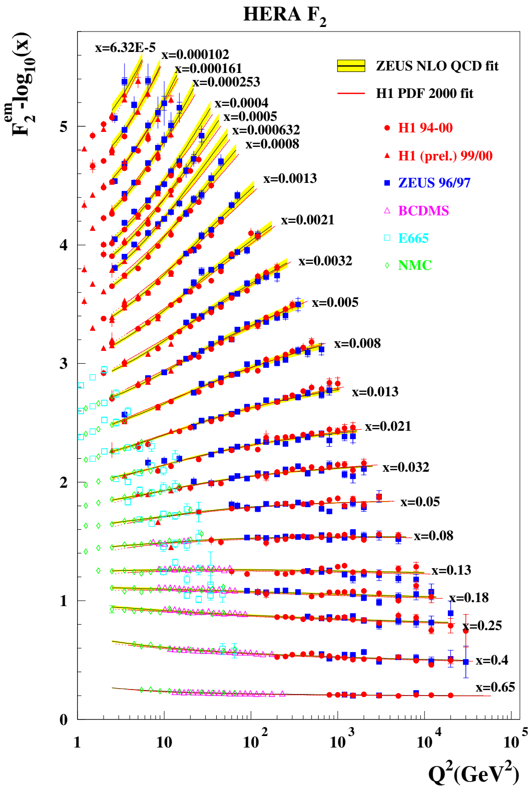
$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{t}{t_0}\right)^{\gamma_{qq}(N,\alpha_S)} ,$$

- We see that dimensionless function D_V, instead of being scale-independent function of x as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale t (hence the name anomalous dimensions).
- For running coupling $\alpha_S(t)$, scaling violation is power-behaved in $\ln t$ rather than t. Using leading-order formula $\alpha_S(t) = 1/b\ln(t/\Lambda^2)$, we find

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d_{qq}(N)}$$

  where $d_{qq}(N) = \gamma^{(0)}_{qq}(N)/2\pi b$.
- Flavour-singlet distribution and quantitative predictions will be discussed later.

Combined data on F2 proton



- Now $d_{qq}(1) = 0$ and $d_{qq}(N) < 0$ for $N \geq 2$. Thus as t increases V *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data:

Flavour singlet combination

- For flavour-singlet combination, define

$$\Sigma = \sum_i (q_i + \bar{q}_i) \; .$$

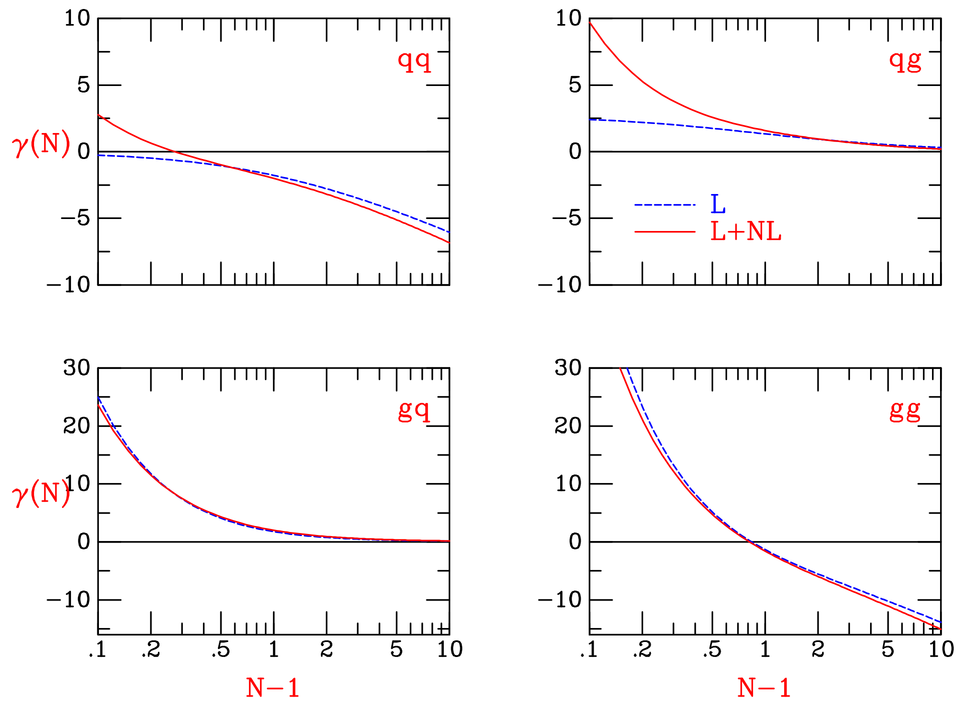
Then we obtain

$$\begin{aligned} t\frac{\partial \Sigma}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{qq} \otimes \Sigma + 2N_f P_{qg} \otimes g\right] \\ t\frac{\partial g}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{gq} \otimes \Sigma + P_{gg} \otimes g\right] \; . \end{aligned}$$

- Thus flavour-singlet quark distribution Σ mixes with gluon distribution g: evolution equation for moments has matrix form

$$t\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{g} \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & 2N_f\gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{g} \end{pmatrix}$$

Anomalous dimension matrix as a function of N.



- Rapid growth at small N in gq and gg elements at lowest order
- $\ln N$ behaviour at large N in qq and gg elements
- NNLO now known
- Singlet anomalous dimension matrix has two real eigenvalues $\gamma_\pm$ given by

$$\gamma_\pm = \frac{1}{2}[\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg} - \gamma_{qq})^2 + 8N_f \gamma_{gq}\gamma_{qg}}] \, .$$

Solution of lowest order DGLAP matrix equation

The reduced DGLAP equation can be written as

$$\frac{d}{du}\begin{pmatrix} \tilde{\Sigma}(u) \\ \tilde{g}(u) \end{pmatrix} = \mathbf{P}\begin{pmatrix} \tilde{\Sigma}(u) \\ \tilde{g}(u) \end{pmatrix}$$

where $u = \frac{1}{2\pi b}\ln\frac{\alpha_S(\mu_0^2)}{\alpha_S(\mu^2)}$

- Define projection operators, $\mathbf{M}_\pm$

$$\mathbf{M}_+ = \frac{1}{\gamma_+ - \gamma_-}\Big[+\mathbf{P} - \gamma_-\mathbf{1}\Big],\ \ \mathbf{M}_- = \frac{1}{\gamma_+ - \gamma_-}\Big[-\mathbf{P} + \gamma_+\mathbf{1}\Big],$$

  where $\mathbf{M}_\pm\mathbf{M}_\pm = \mathbf{M}_\pm, \mathbf{M}_+\mathbf{M}_- = \mathbf{M}_-\mathbf{M}_+ = \mathbf{0}$, $\mathbf{M}_+ + \mathbf{M}_- = \mathbf{1}$ and

$$\mathbf{P} = \gamma_+\mathbf{M}_+ + \gamma_-\mathbf{M}_-$$

- The solution is

$$\begin{pmatrix} \tilde{\Sigma}(u) \\ \tilde{g}(u) \end{pmatrix} = \Big[\mathbf{M}_+ \exp(\gamma_+ u) + \mathbf{M}_- \exp(\gamma_- u)\Big]\begin{pmatrix} \tilde{\Sigma}(0) \\ \tilde{g}(0) \end{pmatrix}$$

Momentum partition vs Q^2

- For second moment

$$
\begin{aligned}
O^{+}(2,t) &= \Sigma(2,t) + g(2,t) \quad \text{with eigenvalue } 0 \,, \\
O^{-}(2,t) &= \Sigma(2,t) - \frac{n_f}{4C_F} g(2,t) \quad \text{with eigenvalue } -\left(\frac{4}{3}C_F + \frac{n_f}{3}\right) \,.
\end{aligned}
$$

O^{+}, corresponds to the total momentum carried by the quarks and gluons, is independent of t. The eigenvector O^{-} vanishes in the limit $t \to \infty$:

$$
O^{-}(2,t) = \left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d^{-}(2)} \to 0, \quad \text{with } \; d^{-}(2) = \frac{\gamma_{-}(2)}{2\pi b} = -\frac{\left(\frac{4}{3}C_F + \frac{1}{3}n_f\right)}{2\pi b} \,,
$$

so that asymptotically we have

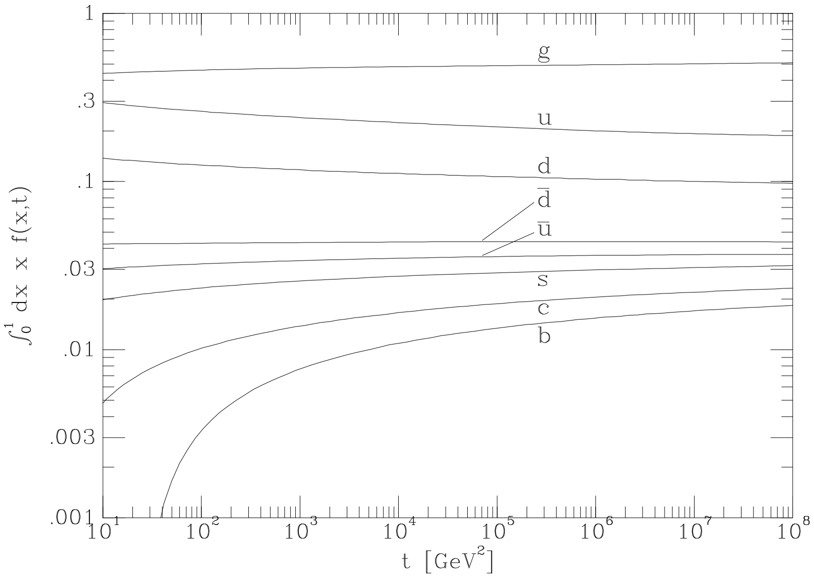
$$
\frac{\Sigma(2,t)}{g(2,t)} \to \frac{n_f}{4C_F} = \frac{3}{16} n_f \,.
$$

Asymptotia is approached slowly

The momentum fractions f_q and f_g in the $\mu^2 = t \to \infty$ limit are therefore

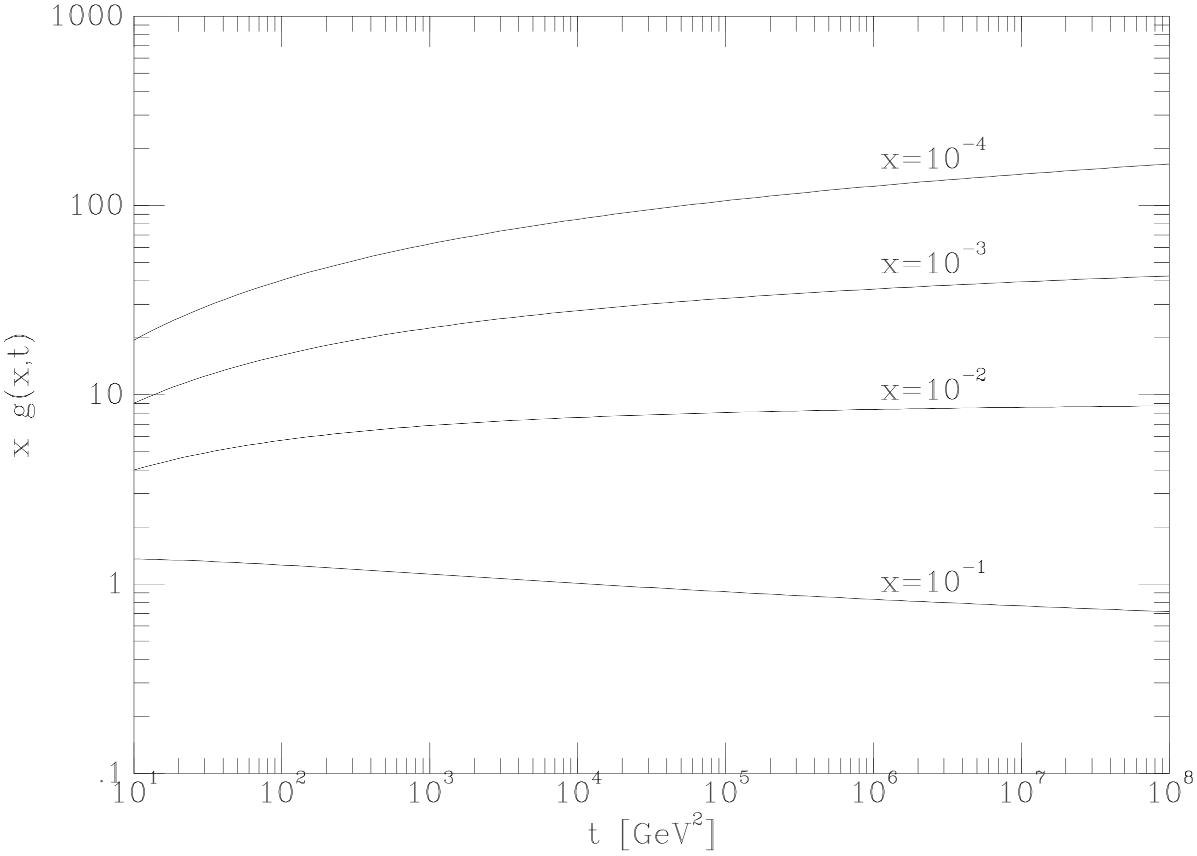
$$f_q = \frac{3n_f}{16+3n_f} \;, \quad f_g = \frac{16}{16+3n_f} \;.$$

- Scaling violation depends logarithmically on Q^2.
- Large variation at low Q^2



Gluon distribution

- Large number of gluons per unit rapidity
- The LHC is a copious source of gluons



Sudakov form factor

- DGLAP equations convenient for evolution of parton distributions. To study structure of final states, slightly different form is useful. Consider again simplified treatment with only one type of branching. Introduce Sudakov form factor:

$$\Delta(t) \equiv \exp\left[-\int_{t_0}^{t} \frac{dt'}{t'} \int dz \, \frac{\alpha_S}{2\pi} \hat{P}(z)\right] ,$$

$$\begin{aligned} t\frac{\partial}{\partial t} D(x,t) &= \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) + \frac{D(x,t)}{\Delta(t)} t\frac{\partial}{\partial t}\Delta(t) , \\ t\frac{\partial}{\partial t}\left(\frac{D}{\Delta}\right) &= \frac{1}{\Delta} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) . \end{aligned}$$

Sudakov form factor

- This is similar to DGLAP, except D replaced by D/Δ and regularized splitting function P replaced by unregularized $\hat{P}$. Integrating,

$$D(x,t) \quad = \quad \Delta(t)D(x,t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t') \ .$$

- This has simple interpretation. First term is contribution from paths that do not branch between scales t_0 and t. Thus Sudakov form factor $\Delta(t)$ is probability of evolving from t_0 to t without branching. Second term is contribution from paths which have their last branching at scale t'. Factor of $\Delta(t)/\Delta(t')$ is probability of evolving from t' to t without branching.

Sudakov form factor

- Generalization to several species of partons straightforward. Species i has Sudakov form factor

$$\Delta_i(t) \equiv \exp\left[-\sum_j \int_{t_0}^{t} \frac{dt'}{t'} \int dz\, \frac{\alpha_S}{2\pi} \hat{P}_{ji}(z)\right] \;,$$

which is probability of it evolving from t_0 to t without branching. Then

$$t\frac{\partial}{\partial t}\left(\frac{D_i}{\Delta_i}\right) = \frac{1}{\Delta_i}\sum_j \int \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}_{ij}(z) D_j(x/z,t) \;.$$

Infrared cutoff

- In DGLAP equation, infrared singularities of splitting functions at $z = 1$ are regularized by plus-prescription. However, in above form we must introduce an explicit infrared cutoff, $z < 1 - \epsilon(t)$. Branchings with z above this range are unresolvable: emitted parton is too soft to detect. Sudakov form factor with this cutoff is probability of evolving from t_0 to t without any resolvable branching.
- Sudakov form factor sums enhanced virtual (parton loop) as well as real (parton emission) contributions. No-branching probability is the sum of virtual and unresolvable real contributions: both are divergent but their sum is finite.
- Infrared cutoff $\epsilon(t)$ depends on what we classify as resolvable emission. For timelike branching, natural resolution limit is given by cutoff on parton virtual mass-squared, $t > t_0$. When parton energies are much larger than virtual masses, transverse momentum in $a \to bc$ is

  $$p_T^2 = z(1-z)p_a^2 - (1-z)p_b^2 - zp_c^2 > 0 \ .$$

  Hence for $p_a^2 = t$ and $p_b^2,\, p_c^2 > t_0$ we require

  $$z(1-z) > t_0/t \ ,$$

  that is,

  $$z,\ 1-z > \epsilon(t) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4t_0/t} \simeq t_0/t \ .$$

- Quark Sudakov form factor is then

$$\Delta_q(t) \simeq \exp\left[-\int_{2t_0}^{t} \frac{dt'}{t'} \int_{t_0/t'}^{1-t_0/t'} dz \frac{\alpha_S}{2\pi} \hat{P}_{qq}(z)\right] .$$

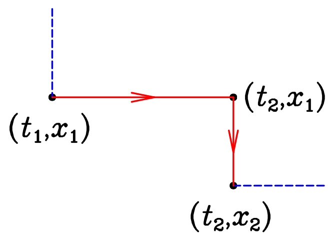
- Careful treatment of running coupling suggests its argument should be $p_T^2 \sim z(1-z)t'$. Then at large t

$$\Delta_q(t) \sim \left(\frac{\alpha_S(t)}{\alpha_S(t_0)}\right)^{p \ln t} ,$$

  ($p =$ a constant), which tends to zero faster than any negative power of t.
- Infrared cutoff discussed here follows from kinematics. We shall see later that QCD dynamics effectively reduces phase space for parton branching, leading to a more restrictive effective cutoff.
- Formulation in terms of Sudakov form factor is well suited to computer implementation, and is basis of “parton shower" Monte Carlo programs.

Monte Carlo method

- Monte Carlo branching algorithm operates as follows: given virtual mass scale and momentum fraction (t_1, x_1) after some step of the evolution, or as initial conditions, it generates values (t_2, x_2) after the next step.



  - ⋆ Since probability of evolving from t_1 to t_2 without branching is $\Delta(t_2)/\Delta(t_1)$, t_2 can be generated with the correct distribution by solving

$$\frac{\Delta(t_2)}{\Delta(t_1)} = \mathcal{R}$$

    where $\mathcal{R}$ is random number (uniform on $[0, 1]$).
  - ⋆ If t_2 is higher than hard process scale Q^2, this means branching has finished.
  - ⋆ Otherwise, generate $z = x_2/x_1$ with distribution proportional to $(\alpha_S/2\pi)P(z)$, where $P(z)$ is appropriate splitting function, by solving

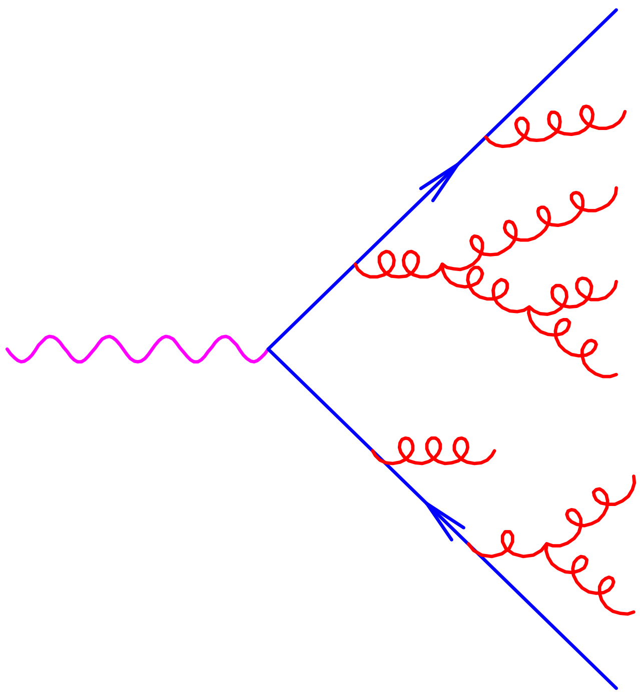
$$\int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_S}{2\pi} P(z) = \mathcal{R}' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_S}{2\pi} P(z)$$

- In DIS, (t_i, x_i) values generated define virtual masses and momentum fractions of exchanged quark, from which momenta of emitted gluons can be computed. Azimuthal emission angles are then generated uniformly in the range $[0, 2\pi]$. More generally, e.g. when exchanged parton is a gluon, azimuths must be generated with polarization angular correlations.
- Each emitted (timelike) parton can itself branch. In that case t evolves downwards towards cutoff value t_0, rather than upwards towards hard process scale Q^2. Probability of evolving downwards without branching between t_1 and t_2 is now given by

$$\frac{\Delta(t_1)}{\Delta(t_2)} = \mathcal{R} \; .$$

Thus branching stops when $\mathcal{R} < \Delta(t_1)$.

Parton Cascade



- Due to successive branching, parton cascade or shower develops. Each outgoing line is source of new cascade, until all outgoing lines have stopped branching. At this stage, which depends on cutoff scale t_0, outgoing partons have to be converted into hadrons via a hadronization model.

Soft gluon emission

- Parton branching formalism discussed so far takes account of collinear enhancements to all orders in PT. There are also soft enhancements: When external line with momentum p and mass m (not necessarily small) emits gluon with momentum q, propagator factor is

$$\frac{1}{(p \pm q)^2 - m^2} = \frac{\pm 1}{2p \cdot q} = \frac{\pm 1}{2\omega E(1 - v\cos\theta)}$$

  where ω is emitted gluon energy, E and v are energy and velocity of parton emitting it, and θ is angle of emission. This diverges as $\omega \to 0$, for any velocity and emission angle.

- Including numerator, soft gluon emission gives a colour factor times universal, spin-independent factor in amplitude

$$F_{\text{soft}} = \frac{p \cdot \epsilon}{p \cdot q}$$

  where ϵ is polarization of emitted gluon.

- For example, emission from quark gives numerator factor $N \cdot \epsilon$, where

$$\begin{aligned} N^\mu &= (\not p + \not q + m)\gamma^\mu u(p) \underset{\omega \to 0}{\longrightarrow} (\gamma^\nu \gamma^\mu p_\nu + \gamma^\mu m) u(p) \\ &= (2p^\mu - \gamma^\mu \not p + \gamma^\mu m) u(p) \;=\; 2p^\mu u(p)\ . \end{aligned}$$

(using Dirac equation for on-mass-shell spinor $u(p)$).

- Universal factor F_{soft} coincides with classical eikonal formula for radiation from current p^μ, valid in long-wavelength limit.
- No soft enhancement of radiation from off-mass-shell internal lines, since associated denominator factor $(p+q)^2 - m^2 \to p^2 - m^2 \neq 0$ as $\omega \to 0$.

- Enhancement factor in amplitude for each external line implies cross section enhancement is sum over all pairs of external lines $\{i,j\}$:

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega}\frac{d\Omega}{2\pi}\frac{\alpha_S}{2\pi}\sum_{i,j} C_{ij}W_{ij}$$

  where $d\Omega$ is element of solid angle for emitted gluon, C_{ij} is a colour factor, and radiation function W_{ij} is given by

$$W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q\, p_j \cdot q} = \frac{1 - v_i v_j \cos\theta_{ij}}{(1 - v_i\cos\theta_{iq})(1 - v_j\cos\theta_{jq})} \ .$$

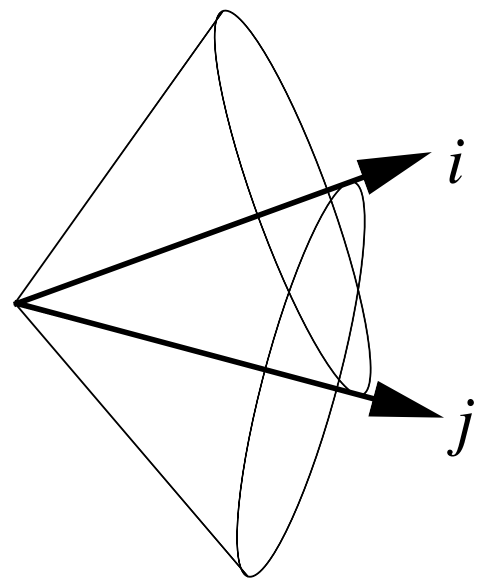
  Colour-weighted sum of radiation functions $C_{ij}W_{ij}$ is antenna pattern of hard process.

- Radiation function can be separated into two parts containing collinear singularities along lines i and j. Consider for simplicity massless particles, $v_{i,j}=1$. Then $W_{ij} = W_{ij}^i + W_{ij}^j$ where

$$W_{ij}^i = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{jq}}\right) \ .$$

- This function has remarkable property of angular ordering. Write angular integration in polar coordinates w.r.t. direction of i, $d\Omega = d\cos\theta_{iq}\, d\phi_{iq}$. Performing azimuthal integration, we find

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^{i} = \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}\text{, otherwise 0.}$$



Thus, after azimuthal averaging, contribution from W_{ij}^{i} is confined to cone, centred on direction of i, extending in angle to direction of j. Similarly, W_{ij}^{j}, averaged over ϕ_{jq}, is confined to cone centred on line j extending to direction of i.

Angular ordering

- To prove angular ordering property, write

$$1-\cos\theta_{jq} = a - b\cos\phi_{iq}$$

where

$$a = 1-\cos\theta_{ij}\cos\theta_{iq}\;, \qquad b = \sin\theta_{ij}\sin\theta_{iq}\;.$$

Defining $z = \exp(i\phi_{iq})$, we have

$$I_{ij}^{i} \equiv \int_0^{2\pi}\frac{d\phi_{iq}}{2\pi}\frac{1}{1-\cos\theta_{jq}} = \frac{1}{i\pi b}\oint\frac{dz}{(z_+-z)(z-z_-)}$$

where z-integration contour is the unit circle and

$$z_\pm = \frac{a}{b}\pm\sqrt{\frac{a^2}{b^2}-1}\;.$$

Now only pole at $z=z_-$ can lie inside unit circle, so

$$I_{ij}^{i} = \sqrt{\frac{1}{a^2-b^2}} = \frac{1}{|\cos\theta_{iq}-\cos\theta_{ij}|}\;.$$

Hence

Coherent branching

- Angular ordering provides basis for coherent parton branching formalism, which includes leading soft gluon enhancements to all orders.
- In place of virtual mass-squared variable t in earlier treatment, use angular variable

$$\zeta = \frac{p_b \cdot p_c}{E_b\, E_c} \simeq 1 - \cos\theta$$

as evolution variable for branching $a \to bc$, and impose angular ordering $\zeta' < \zeta$ for successive branchings. Iterative formula for n-parton emission becomes

$$d\sigma_{n+1} = d\sigma_n \frac{d\zeta}{\zeta} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z) \; .$$

Coherent branching

- In place of virtual mass-squared cutoff t_0, must use angular cutoff ζ_0 for coherent branching. This is to some extent arbitrary, depending on how we classify emission as unresolvable. Simplest choice is

$$\zeta_0 = t_0/E^2$$

for parton of energy E.

- For radiation from particle i with finite mass-squared t_0, radiation function becomes

$$\omega^2 \left(\frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} - \frac{p_i^2}{(p_i \cdot q)^2} \right) \simeq \frac{1}{\zeta} \left(1 - \frac{t_0}{E^2 \zeta} \right) ,$$

so angular distribution of radiation is cut off at $\zeta = t_0/E^2$. Thus t_0 can still be interpreted as minimum virtual mass-squared.

- With this cutoff, most convenient definition of evolution variable is not ζ itself but rather

$$\tilde{t} = E^2 \zeta \geq t_0 \ .$$

Coherent branching

Angular ordering condition $\zeta_b, \zeta_c < \zeta_a$ for timelike branching $a \to bc$ (a outgoing) becomes

$$\tilde{t}_b < z^2\tilde{t}\,, \quad \tilde{t}_c < (1-z)^2\tilde{t}$$

where $\tilde{t} = \tilde{t}_a$ and $z = E_b/E_a$. Thus cutoff on z becomes

$$\sqrt{t_0/\tilde{t}} < z < 1 - \sqrt{t_0/\tilde{t}}\,.$$

- Neglecting masses of b and c, virtual mass-squared of a and transverse momentum of branching are

  $$t = z(1-z)\tilde{t}\,, \quad p_t^2 = z^2(1-z)^2\tilde{t}\,.$$

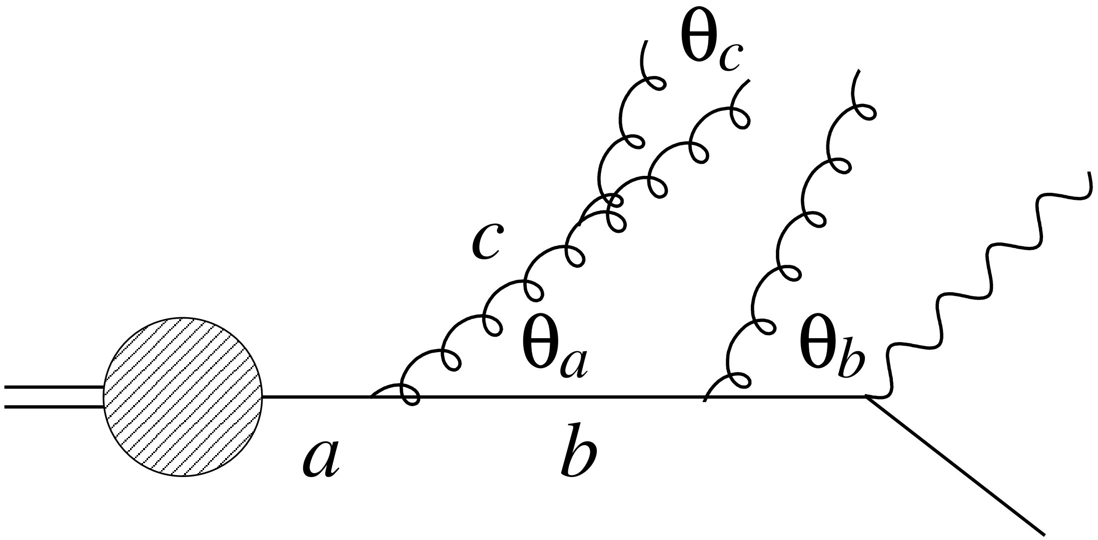
  Thus for coherent branching Sudakov form factor of quark becomes

  $$\tilde{\Delta}_q(\tilde{t}) = \exp\left[-\int_{4t_0}^{\tilde{t}} \frac{dt'}{t'} \int_{\sqrt{t_0/t'}}^{1-\sqrt{t_0/t'}} \frac{dz}{2\pi} \alpha_S(z^2(1-z)^2 t')\hat{P}_{qq}(z)\right]$$

  At large $\tilde{t}$ this falls more slowly than form factor without coherence, due to the suppression of soft gluon emission by angular ordering.

Coherent branching

- Note that for spacelike branching $a \to bc$ (a incoming, b spacelike), angular ordering condition is



$$\theta_b > \theta_a > \theta_c \ ,$$

and so for $z = E_b/E_a$ we now have

$$\tilde{t}_b > z^2 \tilde{t}_a \ , \quad \tilde{t}_c < (1-z)^2 \tilde{t}_a \ .$$

- Thus we can have either $\tilde{t}_b > \tilde{t}_a$ or $\tilde{t}_b < \tilde{t}_a$, especially at small z — spacelike branching becomes disordered at small x.

Recap

- Parton evolution can be represented as a branching process from higher values of x
- DGLAP equation predicts growth at small x and shrinkage at large x with increasing Q^2.
- The Sudakov form factor $\Delta(t)$ is the probability of evolving from t_0 to t without branching.
- branching from (t_1, x_1) to (t_2, x_2) with the right probability can be performed with by choosing three random numbers, (t, x, ϕ)
- Branching is subject to an angular ordering constraint. Large angle emission is dynamically suppressed.