0.1 Dimensional Regularisation

In the intermediate stages of the calculation we must introduce some regularisation procedure to control these divergences. The most effective regulator is the method of dimensional regularisation which continues the dimension of space-time to $d = 4-2\epsilon$ dimensions. This method of regularisation has the advantage that the Ward Identities of the theory are preserved at all stages of the calculation. Integrals over loop momenta are performed in d dimensions with the help of the following formula,

$$\int \frac{d^d k}{(2\pi)^d} \frac{(-k^2)^r}{\Big[-k^2 + C - i\varepsilon \Big]^m} =$$
$$\frac{i(4\pi)^\epsilon}{16\pi^2} \left[C - i\varepsilon\right]^{2+r-m-\epsilon} \frac{\Gamma(r+d/2)}{\Gamma(d/2)} \frac{\Gamma(m-r-2+\epsilon)}{\Gamma(m)}. \tag{1}$$

To demonstrate Eq. (1), we first perform a Wick rotation of the k_0 contour

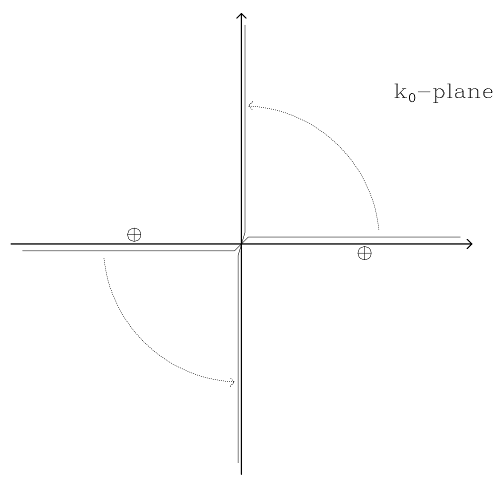

Figure 1: Wick rotation in the complex k_0 plane

anti-clockwise. This is dictated by the $i\varepsilon$ prescription, since for real C the poles coming from the denominator of Eq. (1) lie in the second and fourth quadrant of the k_0 complex plane as shown in Fig. 1. Thus by anti-clockwise rotation we encounter no poles. After rotation by an angle $\pi/2$, the k_0 integral runs along the imaginary axis in the k_0 plane, $(-i\infty < k_0 < i\infty)$. In order to deal only with real quantities we make the substitution $k_0 = i\kappa_d, k_j = \kappa_j$ for all $j \neq 0$ and introduce $|\kappa| = \sqrt{\kappa_1^2 + \kappa_2^2 \ldots + \kappa_d^2}$. We obtain a d-dimensional Euclidean integral which may be written as,

$$\int d^d\kappa\, f(\kappa^2) = \int d|\kappa|\, f(\kappa^2)\, |\kappa|^{d-1} \sin^{d-2}\theta_{d-1} \sin^{d-3}\theta_{d-2} \ldots \times \sin\theta_2\, d\theta_{d-1} d\theta_{d-2} \ldots d\theta_2 d\theta_1. \quad (2)$$

This formula is best proved by induction. The range of the angular integrals is $0 \leq \theta_i \leq \pi$ except for $0 \leq \theta_1 \leq 2\pi$. The angular integrations, which only give an overall factor, can be performed using

$$\int_0^\pi d\theta\, \sin^d\theta = \sqrt{\pi} \frac{\Gamma\Big(\frac{(d+1)}{2}\Big)}{\Gamma\Big(\frac{(d+2)}{2}\Big)}. \quad (3)$$

We therefore find that the left hand side of Eq. (1) can be written as,

$$\frac{2i}{(4\pi)^{d/2}\Gamma\Big(d/2\Big)} \int_0^\infty d|\kappa|\, \frac{|\kappa|^{d+2r-1}}{\Big[\kappa^2 + C\Big]^m}. \quad (4)$$

This last integral can be reduced to a Beta function, (see Table 2)

$$\int_0^\infty dx \frac{x^s}{\Big[x^2 + C\Big]^m} = \frac{\Gamma\big(\frac{(s+1)}{2}\big)}{2} \frac{\Gamma\big(m - s/2 - 1/2\big)}{\Gamma\big(m\big)} C^{s/2+1/2-m} \quad (5)$$

which demonstrates Eq. (1).

In a similar vein to Eq. (2) for phase space integrals we may write

$$\begin{aligned} \int d^{d-1}k &= \frac{2\pi^{1-\epsilon}}{\sqrt{\pi}} \frac{1}{\Gamma(\frac{1}{2} - \epsilon)} \int d|\vec{k}|\, |\vec{k}|^{d-2} \int d\theta_1\, \sin^{d-3}\theta_1 \int d\theta_2\, \sin^{d-4}\theta_2 \\ &= \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \int d|\vec{k}|\, |\vec{k}|^{2-2\epsilon} \int_{-1}^1 dx\, (1-x^2)^{-\epsilon} \\ &= \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \int d|\vec{k}|\, |\vec{k}|^{2-2\epsilon} \frac{2}{4^\epsilon} \int_0^1 dy\, y^{-\epsilon}\, (1-y)^{-\epsilon} \end{aligned} \quad (6)$$

where $x = \cos\theta_1, y = \frac{1}{2}(1+x)$. In the last two equations we have performed the θ_2 integration. In manipulation of these formulae, we often need to use the doubling identity, which in this context takes the form

$$\Gamma(\frac{1}{2} - \epsilon) = 4^\epsilon \sqrt{\pi} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \quad (7)$$

Feynman parameter identities are also useful; we have

$$\frac{1}{A^\alpha\, B^\beta \cdots E^\epsilon} = \frac{\Gamma(\alpha + \beta + \cdots \epsilon)}{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\epsilon)}$$

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\ g^{\mu\nu}\mathrm{I}$ $\gamma^\mu\gamma_\mu = g^\mu_\mu\ \mathrm{I} = d\ \mathrm{I}$ $\gamma^\mu\gamma^\alpha\gamma_\mu = -2\ (1-\epsilon)\ \gamma^\alpha$ $\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4\ g^{\alpha\beta}\mathrm{I} - 2\epsilon\ \gamma^\alpha\gamma^\beta$ $\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\rho\gamma_\mu = -2\ \gamma^\rho\gamma^\beta\gamma^\alpha + 2\epsilon\ \gamma^\alpha\gamma^\beta\gamma^\rho$
$\mathrm{Tr}\ \mathrm{I} = 4$ $\mathrm{Tr}\ \gamma^\mu\gamma^\nu = 4\ g^{\mu\nu}$ $\mathrm{Tr}\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\big(g^{\mu\nu}g^{\rho\sigma} + g^{\nu\rho}g^{\mu\sigma} - g^{\mu\rho}g^{\nu\sigma}\big)$

Table 1: Gamma matrix identities in $d = 4 - 2\epsilon$ dimensions.

$$\begin{aligned} &\times \quad \int_0^1 dx\ dy \cdots dz\ \delta(1 - x - y \cdots - z) \\ &\times \quad \frac{x^{\alpha-1}\ y^{\beta-1} \cdots z^{\epsilon-1}}{(Ax + By + \cdots + Ez)^{\alpha+\beta+\cdots+\epsilon}} \end{aligned} \tag{8}$$

When calculating the two, three and four point functions of the quark, gluon and ghost fields the ultraviolet divergences of the theory appear as poles in ϵ. In the mimimal subtraction (MS) renormalisation scheme one chooses the various Z's of the theory in such a way that the poles are all cancelled. In one loop this leads to the renormalisation constants given in Table 3.

Note that the renormalisation constants depend on the gauge parameter. The scheme is called minimal because the renormalisation constants of the theory contain only the pole parts.

$\Gamma(z) = \int_0^\infty \ dt\ e^{-t} t^{z-1}$
$z\Gamma(z) = \Gamma(z+1)$
$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z+\frac{1}{2})$
$\Gamma(n+1) = n!$ for n a positive integer
$\Gamma(1) = 1, \ \ \Gamma(\frac{1}{2}) = \sqrt{\pi}$
$\Gamma\,'(1) = -\gamma_E, \ \ \gamma_E \approx 0.577215$
$\Gamma\,''(1) = \gamma_E^2 + \frac{\pi^2}{6}$
$B(a,b) = \int_0^1 dx\ x^{a-1}(1-x)^{b-1}$
$B(a,b) = \int_0^\infty dt\ \frac{t^{a-1}}{(1+t)^{a+b}}$ for Re $a, b > 0$
$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Table 2: Useful properties of the Γ and related functions

Z_3	$1+\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[N_c(\frac{13}{6}-\frac{\lambda}{2})-\frac{4}{3}n_fT_R\Big]$
Z_1	$1+\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[N_c(\frac{17}{12}-\frac{3\lambda}{4})-\frac{4}{3}n_fT_R\Big]$
Z_4	$1+\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[N_c(\frac{2}{3}-\lambda)-\frac{4}{3}n_fT_R\Big]$
$\tilde{Z}_3$	$1+\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[N_c(\frac{3}{4}-\frac{\lambda}{4})\Big]$
$\tilde{Z}_1$	$1-\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[N_c\frac{\lambda}{2}\Big]$
Z_2^F	$1-\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[C_F\lambda\Big]$
Z_1^F	$1-\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[N_c(\frac{3}{4}+\frac{\lambda}{4})+C_F\lambda\Big]$
Z_m	$1-\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[C_F3\Big]$
Z_g	$1-\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\Big[N_c\frac{11}{6}-n_fT_R\frac{2}{3}\Big]$

Table 3: Minimal subtraction renormalisation constants in a general covariant gauge at one loop order.