# *Marching orders for QCD*

## *UK Annual theory meeting, December 19-21, 2005.*

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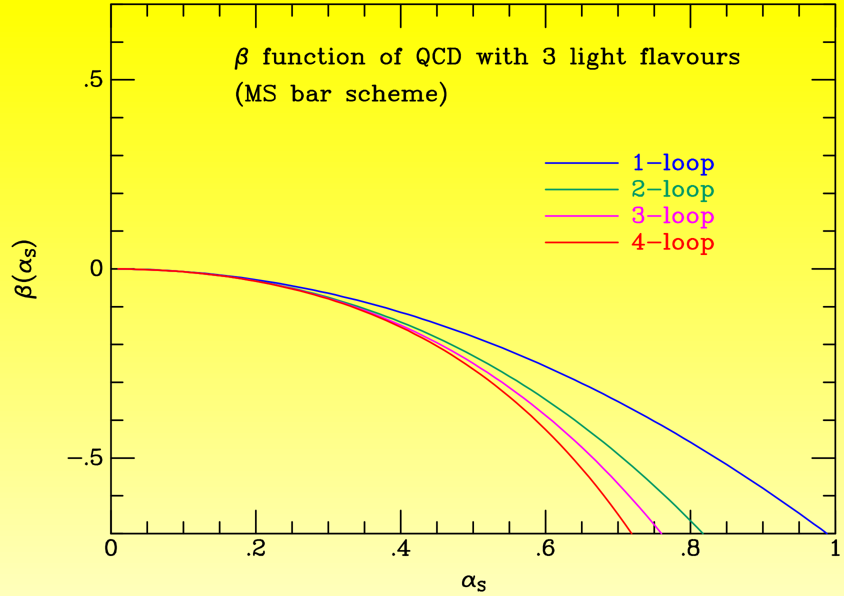
Slides available at http://theory.fnal.gov/people/ellis/Talks/

# $\beta$ function of QCD.

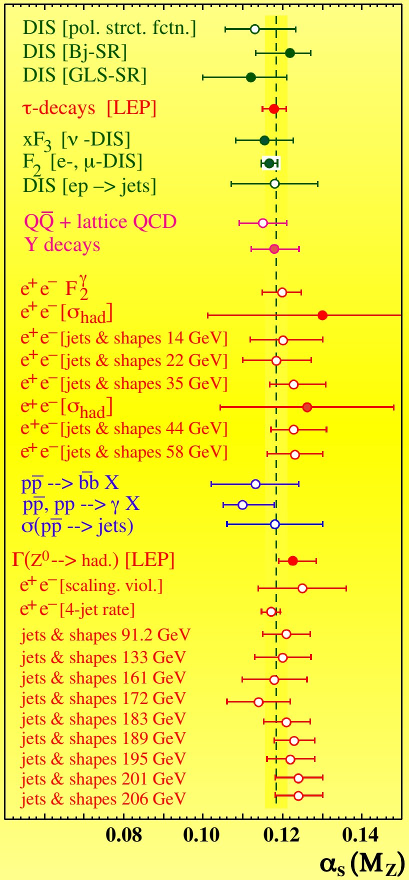
The $\beta$-function of QCD is negative. Terms up to $\mathcal{O}(\alpha_S^5)$ are known.

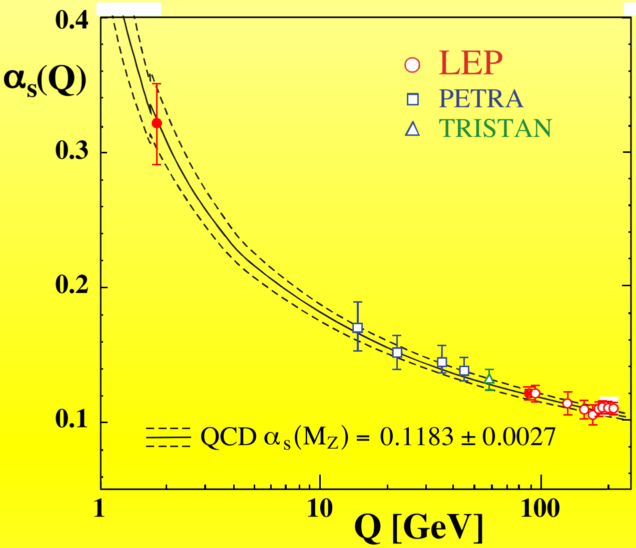
- $\alpha_S^2$: Gross and Wilczek ; Politzer
- $\alpha_S^3$: W. E. Caswell; D. R. T. Jones; E. Egorian and O. V. Tarasov
- $\alpha_S^4$: O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov; S. A. Larin and J. A. M. Vermaseren
- $\alpha_S^5$: T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin



# Current experimental results on $\alpha_S$

Bethke,hep-ph/0407021




$$\alpha_S(M_Z) =$$
$$0.1182 \pm 0.0027, \overline{\text{MS}}, \text{NNLO}$$

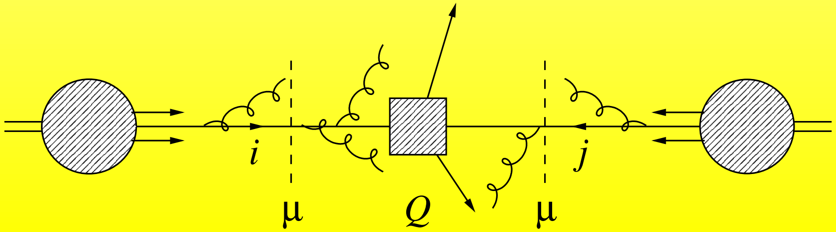
- The decrease of $\alpha_S$ is quite slow – as the inverse power of a logarithm.
- $\alpha_S$ is large at current scales.
- Higher order corrections are important.

# The challenge

- The challenge is to provide the most accurate information possible to experimenters working at the Tevatron and the LHC.
- Proton (anti)proton collisions give rise to a rich event structure.
- Complexity of the events will increase as we pass from the Tevatron to the LHC.
- The goals
  - ⋆ To provide physics software tools which are both flexible and give the most accurate representations of the underlying theories.
  - ⋆ To discover new efficient ways of calculating in perturbative QCD.

# Hadron-hadron processes

- In hard hadron-hadron scattering, constituent partons from each incoming hadron interact at short distance (large momentum transfer $Q^2$).




- Form of cross section is

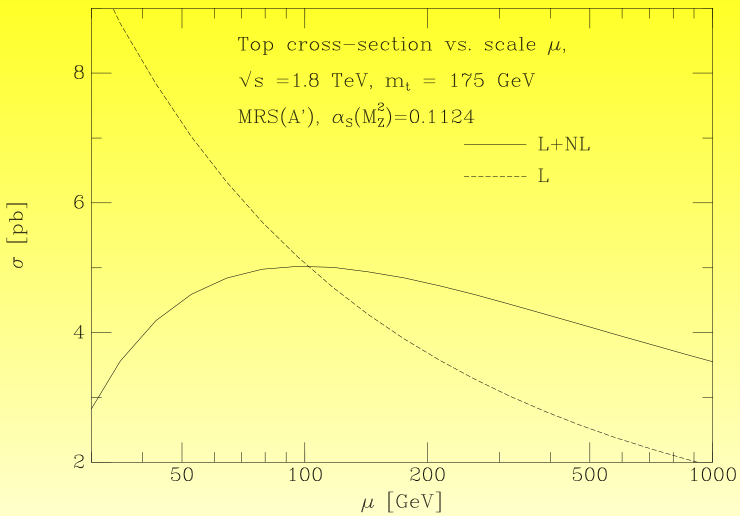
$$\frac{d\sigma}{dX} = \sum_{i,j}\sum_{\tilde{X}}\int dx_1 dx_2\, f_i(x_1,\mu^2) f_j(x_2,\mu^2)$$
$$\times \quad \hat{\sigma}_{ij}^{\tilde{X}}(\alpha_S(\mu^2), Q^2, \mu^2)\, F(\tilde{X}\to X, \mu^2)$$

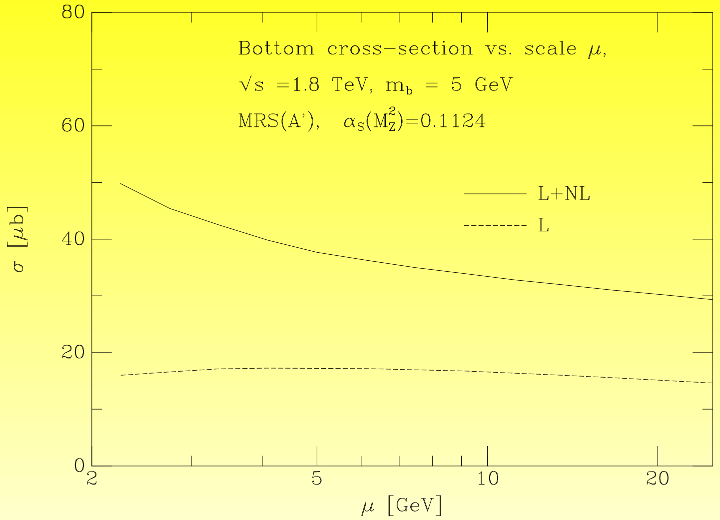
where $\mu^2$ is factorization scale, $\hat{\sigma}_{ij}$ is subprocess cross section for parton types $i, j$ and $X$ represents the hadronic final state.

# Hadron-hadron processes II

- $\hat{\sigma}_{ij}$ is calculable as a perturbation series in $\alpha_S$.
- The physical predictions are invariant under changes of $\mu$ at the appropriate order in perturbation theory. Thus if we have performed a calculation to $O(\alpha_S^3)$, variations of the scale $\mu$ will lead to corrections of $O(\alpha_S^4)$,

$$\mu^2 \frac{d}{d\mu^2} \sigma = O(\alpha_S^4).$$



# *Approaches to the calculation of $\hat{\sigma}$*

- LO
  - ⋆ Automatic calculation of tree graphs (Madgraph/Helas, Alpgen, CompHEP, . . .)
  - ⋆ LO + parton shower
  - ⋆ New analytic techniques
- NLO
  - ⋆ Analytic techniques for loop diagrams
  - ⋆ Parton level Monte Carlo (MCFM, NLOJET++, . . .)
  - ⋆ Numerical techniques for loop diagrams
  - ⋆ NLO + parton shower (MC@NLO)
- NNLO
  - ⋆ a few (mostly) inclusive results are known

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- NNLO
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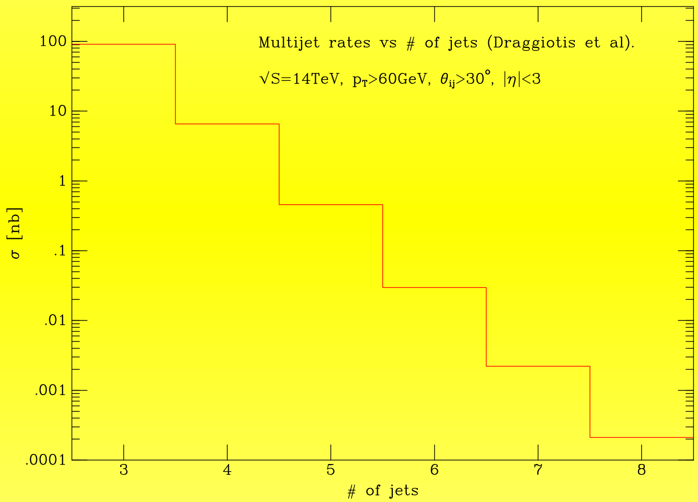
# *Multijet rates using tree graphs*

- Calculation of tree graphs using off-shell recurrence relations is a solved problem Berends, Giele.



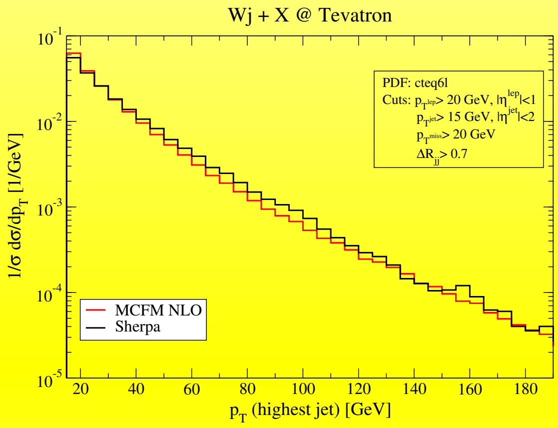

Draggiotis et al

- At $10^{33}$ cm$^{-2}$s$^{-1}$, left hand scale gives events per second
- $g = 1$
- Similar calculations are possible with other programs Madgraph, Alpgen, COMPHEP, ...

# The role of tree graphs

- Evaluation of tree graphs with $\leq 9$ partons is achievable in a moderate amount of CPU time.
- For example, for $W, Z$ + n jets at tree graph level.
  Madgraph II can generate processes with $\leq$ 9 external particles (madgraph.hep.uiuc.edu)
  Vecbos, W-boson plus up to 4 jets or a Z-boson plus up to 3 jets (theory.fnal.gov/people/giele/vecbos.html)
  Alpgen, W,Z + up to 6 jets.
- Problems with tree graphs
  - ⋆ Overall normalization is uncertain. $W$+4 jets occurs at O($\alpha_S^4$). If scale uncertainty changes $\alpha_S$ by 10%, this leads to 40% uncertainty in cross section.
  - ⋆ Sometimes a new parton process appears at NLO, leading to large change in shapes. (e.g., gluons at the LHC).
  - ⋆ Require a procedure to combine with parton showers.

# *Combining Matrix elements and parton showers*




- Divide phase space into two regions – region I for jet production modeled by the appropriate matrix element, region II for jet evolution modeled by the parton shower.
- Procedure to cancel the leading dependence on separation parameter (CKKW)

- $p_T$ spectrum of the hardest jet in inclusive $W$+1 jet, using Matrix element improved showering scheme.
- Agreement in shape between exact NLO calculation and ME improved shower (SHERPA).

F. Krauss et al, hep-ph/0409106

# *Spinor notation*

- Denote spinor for lightlike vectors as follows:-

$$\begin{aligned} |k+\rangle &= \text{right-handed spinor for massless vector } k \\ |k-\rangle &= \text{left-handed spinor for massless vector } k \end{aligned}$$

- Polarization vectors are given by ($q \equiv$ gauge choice)

$$\varepsilon_\mu^+(k) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\langle qk \rangle}, \quad \varepsilon_\mu^-(k) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}[kq]}$$

- Obeys all the requirements of a polarization vector

$$\varepsilon_i^2 = 0, \quad k \cdot \varepsilon(k) = 0, \quad q \cdot \varepsilon(k) = 0, \quad \varepsilon^+ \cdot \varepsilon^- = -1$$

- Equivalent notations

$$\epsilon^{ab}\lambda_{ja}\lambda_{lb} \equiv \langle jl \rangle \equiv \langle k_j^- | k_l^+ \rangle = \sqrt{2k_j \cdot k_l}\, e^{i\phi}$$

$$\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{j\dot{a}}\tilde{\lambda}_{l\dot{b}} \equiv [jl] \equiv \langle k_j^+ | k_l^- \rangle = -\sqrt{2k_j \cdot k_l}\, e^{-i\phi}$$

# *MHV amplitudes – 5 gluon amplitude*

- Decompose gluonic amplitude into color-ordered sub-amplitudes

$$A = \text{Tr}\{t^{a1}t^{a2}t^{a3}t^{a4}t^{a5}\}m(1,2,3,4,5) + 23 \text{ permutations}$$

- Two of the color stripped amplitudes vanish

$$\begin{aligned} m(g_1^+, g_2^+, g_3^+, g_4^+, g_5^+) &= 0 \\ m(g_1^-, g_2^+, g_3^+, g_4^+, g_5^+) &= 0 \end{aligned}$$

- The maximal helicity violating 5 gluon amplitude

$$m(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$\langle ij\rangle, [ij]$ useful because QCD amplitudes have square root singularities.

# *MHV amplitudes*

Parke and Taylor, Berends and Giele

- The generalization to the case with two contiguous positive helicity gluons and $n-2$ negative gluons is

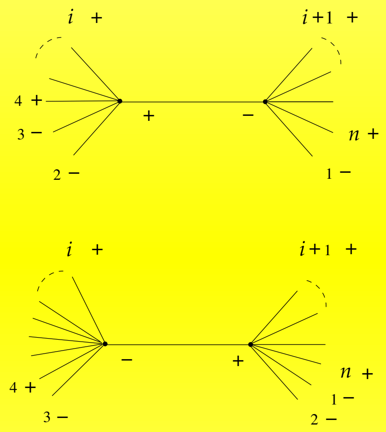
$$m(g_1^-, g_2^-, g_3^+, \ldots g_n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}$$

- Remember $\langle ij\rangle$ are the spinor products $\sim \sqrt{(2p_i \cdot p_j)}$
- Intuition from twistor space leads to two advances using spinors:
  - ⋆ Building more complicated amplitudes using effective MHV vertices.
  - ⋆ BCFW On-shell recursion relations.

# *MHV calculus*

Cachazo,Svrcek,Witten

- Use MHV amplitudes as effective vertices to build more complicated amplitudes




- Obtain simple expressions for tree amplitudes in terms of spinor products
- Individual terms in the expressions for tree amplitudes contain spurious poles which cancel in the sum. These may compromise the utility of the expressions for numerical evaluation.

# *On-shell recursion*

Britto et al, hep-th/0501052

- Perform shift of the momenta conserving masslessness and overall momentum conservation

$$p_1^\mu \to \hat{p}_1^\mu = p_1^\mu - \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle, \quad p_n^\mu \to \hat{p}_n^\mu = p_n^\mu + \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle$$

$$\hat{P}_{1,k}^2 = P_{1,k}^2 - z\langle 1^-|\not{P}_{i,k}|n^-\rangle, \quad P_{1,k} = \sum_{j=1}^{k} p_j$$

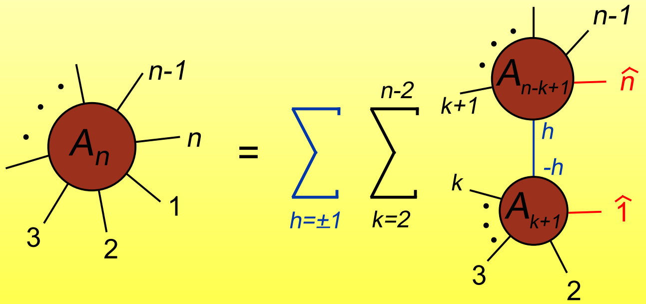
- For each partition of momenta we obtain a pole $z_\alpha$

$$z_\alpha = \frac{P_{1,k}^2}{\langle 1^-|\not{P}_{i,k}|n^-\rangle}$$

- If the tree amplitudes, $A(z)$, vanish for $z \to \infty$ we can close the contour on the integral $\frac{1}{2\pi i}\oint_c \frac{dz}{z} A(z)$

# *On-shell recursion*

$$A(0) = -\sum_{\text{poles } \alpha} \text{Residue}_{z=z_\alpha} \frac{A(z)}{z}$$




$$A_n^{\text{tree}}(1,2,\ldots,n) = \sum_{h=\pm 1}\sum_{k=2}^{n-2} A_{k+1}^{\text{tree}}(\hat{1},2,\ldots,k,-\hat{P}_{1,k}^{-h})\frac{i}{P_{1,k}^2}$$
$$\times \quad A_{n-k+1}^{\text{tree}}(\hat{P}_{1,k}^{h},k+1,\ldots,n-1,\hat{n}).$$

- Thus for the six gluon amplitude (220 diagrams) we have

$$A_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{i}{\langle 2^-|(6+1)|5^-\rangle}$$
$$\times \quad \left[\frac{(\langle 6^-|(1+2)|3^-\rangle)^3}{\langle 61\rangle\langle 12\rangle[34][45]s_{612}} + \frac{(\langle 4^-|(5+6)|1^-\rangle)^3}{\langle 23\rangle\langle 34\rangle[56][61]s_{561}}\right]$$

- notice unphysical singularities when the sum if 1+6 is a linear combination of 2 and 5.

# *MHV outlook*

- Lead to beautiful results for gauge theory amplitudes; however the evaluation of pure gluon tree graphs is a numerically solved problem, (Berends-Giele recursion).
- Elegance of results relies on the introduction spurious singularities; these can have a bearing on their numerical utility.
- So far impact on real phenomenology limited; simple tree graph results for Higgs+5 parton amplitudes Dixon et al, Badger et al
- Extension to loops is the next frontier.

# NLO

# *Why NLO?*

The benefits of higher order calculations are:-

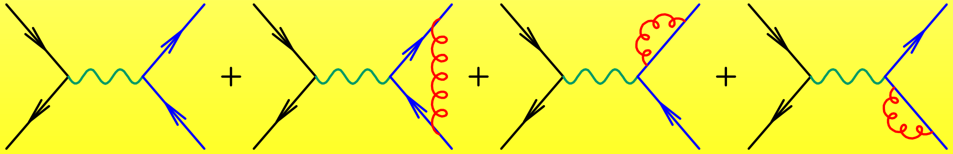
- Less sensitivity to unphysical input scales (eg. renormalization scale)
  - ⋆ First prediction of normalization of observables at NLO
  - ⋆ More accurate estimates of backgrounds for new physics searches.
  - ⋆ Confidence that cross-sections are under control for precision measurements
- More physics
  - ⋆ Jet merging
  - ⋆ Initial state radiation
  - ⋆ More species of incoming partons enter at NLO
- It represents the first step for other techniques matching with resummed calculations, eg. NLO parton showers

# *NLO calculation*

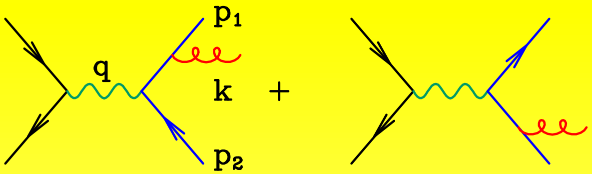
- Ingredients in a NLO calculation are
  - ⋆ Born level amplitude
  - ⋆ Real contribution: Addition of one extra parton to Born level process
  - ⋆ Virtual contribution: Interference of one-loop amplitude with Born amplitude
- Real and virtual separately contain singularities from the soft and collinear regions which cancel in the sum.
- Current stumbling block is the calculation of virtual one loop diagrams
  - ⋆ Calculation of one loop amplitude rapidly becomes complicated as number of partons increases.
  - ⋆ Especially true as we go beyond the most symmetric cases with all gluons.

# *Example: $e^+e^-$ total rate*

- Consider the $O(\alpha_S)$ corrections to total $e^+e^- \to q\bar{q}$ rate.



(a)




(b)

- Virtual gluon contributions (a): using dimensional regularization
- Soft and collinear singularities are regulated, appearing instead as poles at $D-4=-2\epsilon$.

$$\sigma^{q\bar{q}} \;=\; 3\sigma_0\,\left\{1+\frac{2\alpha_S}{3\pi}\,H(\epsilon)\,\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\pi^2+\mathcal{O}(\epsilon)\right]\right\}\;.$$

# *Example: $e^+e^-$ total rate*

- Real contributions integrated over unobserved gluon

$$\sigma^{q\bar{q}g} \ = \ 2\sigma_0 \frac{\alpha_S}{\pi} \ H(\epsilon) \ \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right] \ .$$

- Adding real and virtual contributions, poles cancel and result is finite as $\epsilon \to 0$. Total cross section is finite (infrared safe).

$$\sigma^{q\bar{q}+q\bar{q}g} \ = \ 3\sigma_0 \ \left\{ 1 + \frac{\alpha_S}{\pi} + \mathcal{O}(\alpha_S^2) \right\}$$

- However the virtual corrections to processes with a larger number of legs, for example, $W^+ \to u\bar{d}gggg$ (relevant for $W$ +4 jets calculation) are not so easily calculated.

# *Pure QCD amplitudes at one loop*

- Four parton processes
  - $\star$ $q\bar{q}q'\bar{q}'$ at one loop — R.K. Ellis, Furman, Haber, Hinchliffe, 1980
  - $\star$ $q\bar{q}gg, gggg$ at one loop — R.K. Ellis, Sexton, 1985
- Five parton processes
  - $\star$ $ggggg$ at one loop — Bern et al, hep-ph/9302280
  - $\star$ $q\bar{q}ggg$ at one loop — Bern et al, hep-ph/9409393
  - $\star$ $q\bar{q}q'\bar{q}'g$ at one loop — Kunszt, hep-ph/9405386
- Six parton processes
  - $\star$ Partial $gggggg$ at one loop (completion expected in 2006)
    Bern et al, hep-ph/0505005,hep-ph/0505055,hep-ph/0507005,hep-ph/0412210
    Britto et al hep-ph/0503132,Bedford et al, hep-th/0412108
  - $\star$ Advances in six parton amplitudes, using all the theoretical tools, cut-constructibility, Susy (Yang-Mills) decomposition, BCFW recursion . . .

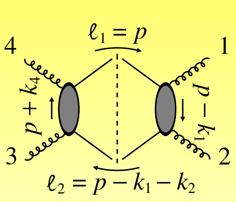
# *Decomposition of six gluon amplitude*

- Six gluon amplitude broken down in to simpler pieces

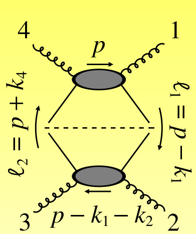
$$\mathcal{A}^{\rm gluon} = \mathcal{A}^{\mathcal{N}=4} - 4\mathcal{A}^{\mathcal{N}=1} + \mathcal{A}^{\rm scalar}$$

  where $\mathcal{A}^{\rm gluon/scalar}$ denotes an amplitude with only a gluon/complex scalar running in the loop.

- Because of their improved ultraviolet behaviour, the supersymmetric pieces of the amplitude are cut-constructible. Full SUSY amplitude determined by the discontinuities, ie tree diagrams



(a)



(b)

$$\int d^n p \frac{2\pi\delta(p^2)2\pi\delta((p-k_1-k_2)^2)}{(p+k_4)^2(p-k_1)^2}$$

$$\rightarrow \int d^n p \frac{1}{(p+k_4)^2 p^2 (p-k_1)^2 (p-k_1-k_2)^2}$$

# *NLO Monte Carlo programs*

- Two programs for 3 jet production at Hadronic colliders
  Kilgore, Giele, hep-ph/9610433, Nagy,hep-ph/0307268
- The virtual corrections to the pure QCD processes are the easiest to calculate but the impact of processes leading to leptons, heavy quarks and missing energy is expected to be the larger.
- Many low multiplicity processes involving vector bosons, top quarks, heavy quarks are included in the the parton level Monte Carlo program MCFM.

# *MCFM overview*

John Campbell and R.K. Ellis

- Parton level cross-sections predicted to NLO in $\alpha_S$

| | |
|---|---|
| $p\bar{p} \to W^\pm/Z$ | $p\bar{p} \to W^+ + W^-$ |
| $p\bar{p} \to W^\pm + Z$ | $p\bar{p} \to Z + Z$ |
| $p\bar{p} \to W^\pm + \gamma$ | $p\bar{p} \to W^\pm/Z + H$ |
| $p\bar{p} \to W^\pm + g^\star\,(\to b\bar{b})$ | $p\bar{p} \to Zb\bar{b}$ |
| $p\bar{p} \to W^\pm/Z +$ 1 jet | $p\bar{p} \to W^\pm/Z +$ 2 jets |
| $p\bar{p}(gg) \to H$ | $p\bar{p}(gg) \to H +$ 1 jet |
| $p\bar{p}(VV) \to H +$ 2 jets | $p\bar{p} \to t + X$ |
| $pp \to t + W$ | |

$\oplus$ less sensitivity to $\mu_R$, $\mu_F$, rates are better normalized, fully differential distributions.

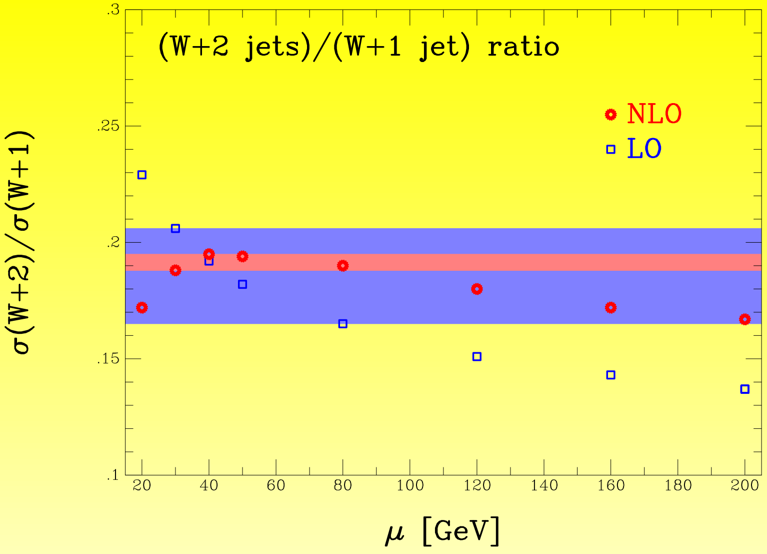
$\ominus$ low particle multiplicity (no showering), no hadronization, hard to model detector effects

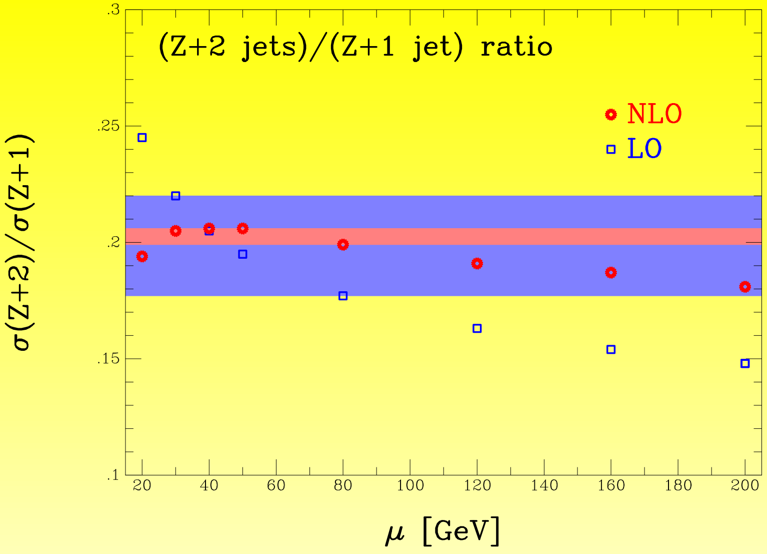
# *MCFM Information*

- Version 4.1 (January 05) available at:
  `http://mcfm.fnal.gov`
- Improvements over previous releases:
  - more processes ($Z + b$, single top, $\ldots$)
  - better user interface
  - support for PDFLIB, Les Houches PDF accord
    ($\longrightarrow$ PDF uncertainties)
  - ntuples as well as histograms
  - unweighted events
  - Pythia/Les Houches generator interface (LO)
  - separate variation of factorization and renormalization scales
  - ‘Behind-the-scenes’ efficiency

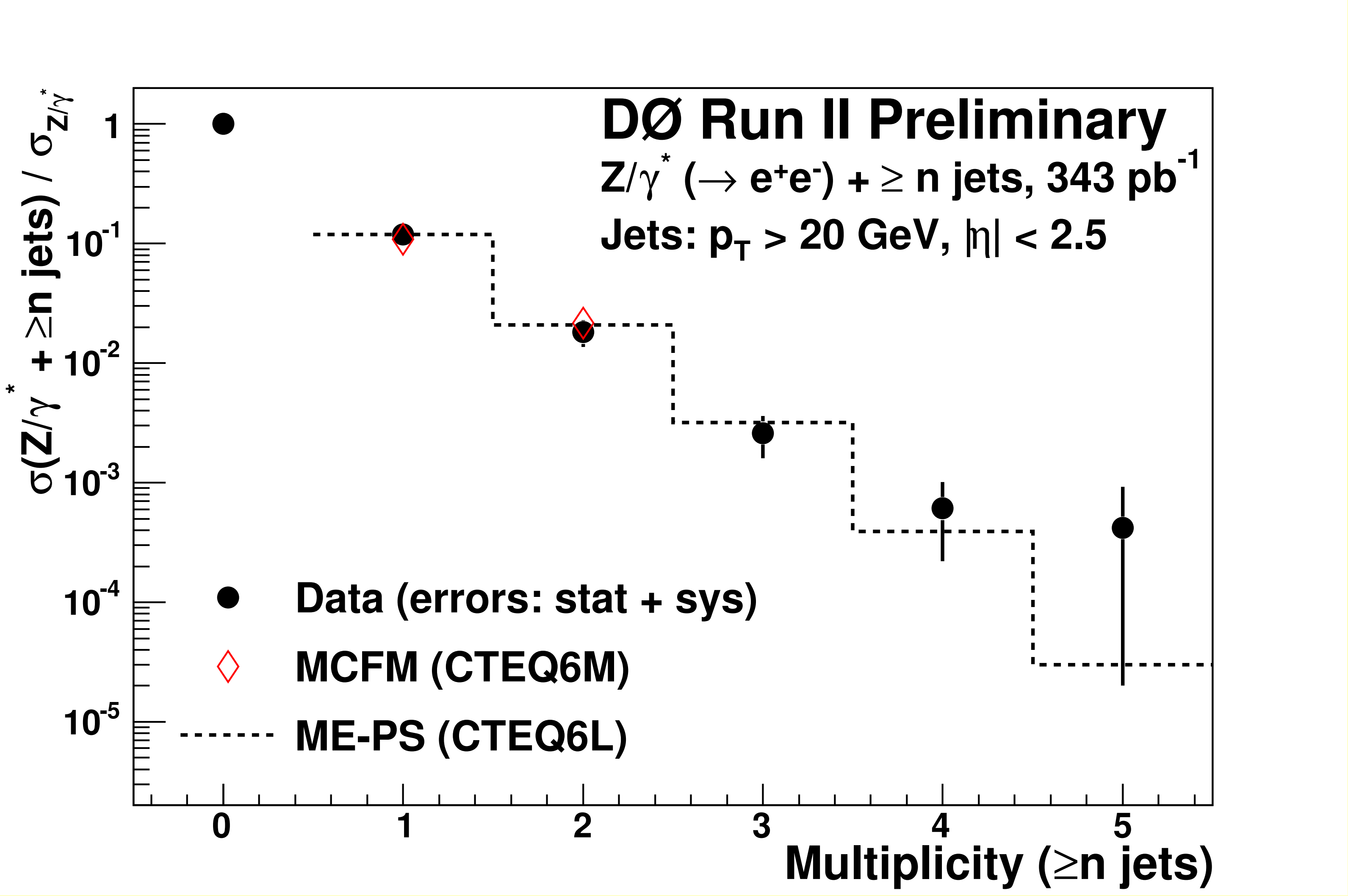
# $W/Z+$ jet cross-sections

- The $W/Z+2$ jet cross-section has been calculated at NLO and should provide an interesting test of QCD (cf. many Run I studies using the $W/Z+1$ jet calculation in DYRAD)
- For instance, the theoretical prediction for the number of events containing $2$ jets divided by the number containing only $1$ is greatly improved.

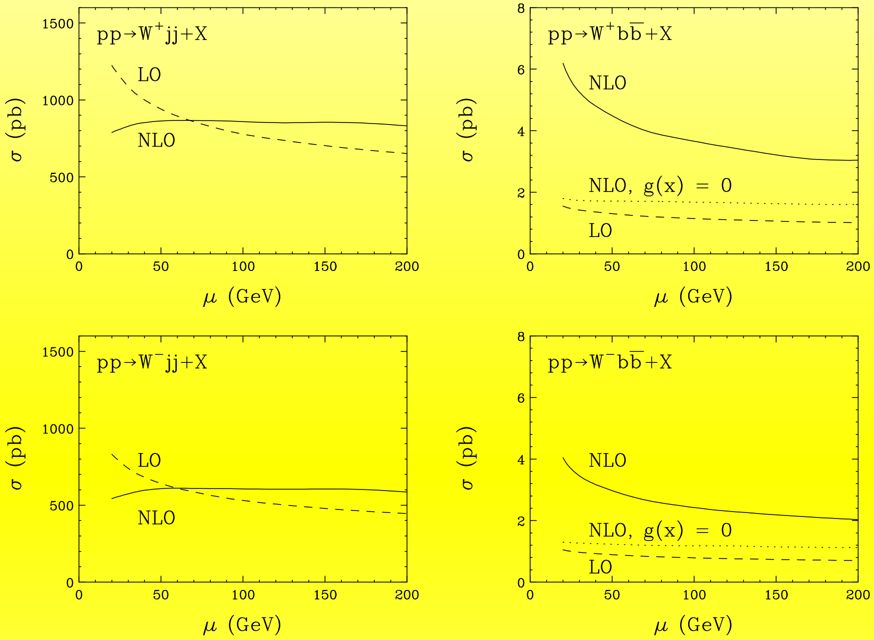


# *D0 data: Z + jets*

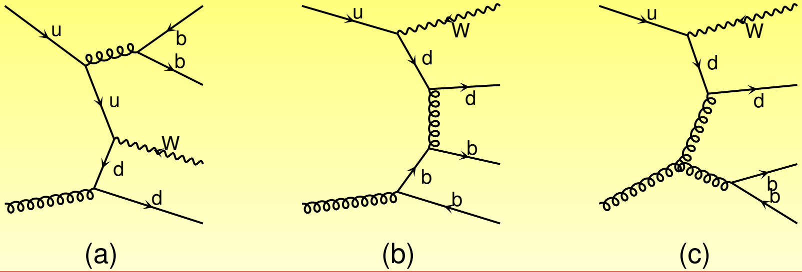


# Jets and heavy flavour at the LHC




- The large gluonic contribution appearing in $Wb\bar{b}$ for the first time at NLO results in a large correction and poor scale dependence.

Diagrams by MadGraph



# An experimenter's wishlist

Run II Monte Carlo Workshop

| Single Boson | Diboson | Triboson | Heavy Flavour |
|---|---|---|---|
| $W+\leq 5j$ | $WW+\leq 5j$ | $WWW+\leq 3j$ | $t\bar{t}+\leq 3j$ |
| $W+b\bar{b}\leq 3j$ | $W+b\bar{b}+\leq 3j$ | $WWW+b\bar{b}+\leq 3j$ | $t\bar{t}+\gamma+\leq 2j$ |
| $W+c\bar{c}\leq 3j$ | $W+c\bar{c}+\leq 3j$ | $WWW+\gamma\gamma+\leq 3j$ | $t\bar{t}+W+\leq 2j$ |
| $Z+\leq 5j$ | $ZZ+\leq 5j$ | $Z\gamma\gamma+\leq 3j$ | $t\bar{t}+Z+\leq 2j$ |
| $Z+b\bar{b}+\leq 3j$ | $Z+b\bar{b}+\leq 3j$ | $ZZZ+\leq 3j$ | $t\bar{t}+H+\leq 2j$ |
| $Z+c\bar{c}+\leq 3j$ | $ZZ+c\bar{c}+\leq 3j$ | $WZZ+\leq 3j$ | $t\bar{b}\leq 2j$ |
| $\gamma+\leq 5j$ | $\gamma\gamma+\leq 5j$ | $ZZZ+\leq 3j$ | $b\bar{b}+\leq 3j$ |
| $\gamma+b\bar{b}\leq 3j$ | $\gamma\gamma+b\bar{b}\leq 3j$ | | |
| $\gamma+c\bar{c}\leq 3j$ | $\gamma\gamma+c\bar{c}\leq 3j$ | | |
| | $WZ+\leq 5j$ | | |
| | $WZ+b\bar{b}\leq 3j$ | | |
| | $WZ+c\bar{c}\leq 3j$ | | |
| | $W\gamma+\leq 3j$ | | |
| | $Z\gamma+\leq 3j$ | | |

# *Tensor one-loop diagrams*

We want to consider tensor integrals of the form

$$I^{\mu_1\cdots\mu_M} = \int \frac{d^D l}{i\pi^{D/2}} \; \frac{l^{\mu_1} \ldots l^{\mu_M}}{d_1 d_2 \ldots d_N}$$

where $d_i = (l + \sum_{j=1}^{j=i} p_j)^2$ are the standard propagator factors.
Passarino and Veltman (1979) wrote a form factor expansion for one-loop integrals, with $M \le N, N \le 4$. For example,

$$\int \frac{d^D l}{i\pi^{D/2}} \frac{l^\mu}{l^2(l+p_1)^2(l+p_1+p_2)^2} = C_1(p_1,p_2)p_1^\mu + C_2(p_1,p_2)p_2^\mu$$

Contracting with $p_1$ and $p_2$ and using the identities

$$l\cdot p_1 = \frac{1}{2}[(l+p_1)^2 - l^2 - p_1^2], l\cdot p_2 = \frac{1}{2}[(l+p_1+p_2)^2 - (l+p_1)^2 - p_2^2 - 2p_1\cdot p_2]$$

# *Historical perspective II*

We derive a linear equation expressing $C_1, C_2$ in terms of scalar integrals

$$\left(\begin{array}{cc} 2p_1\cdot p_1 & 2p_1\cdot p_2 \\ 2p_2\cdot p_1 & 2p_2\cdot p_2 \end{array}\right)\left(\begin{array}{c} C_1 \\ C_2 \end{array}\right)=\left(\begin{array}{c} R_1 \\ R_2 \end{array}\right)$$

where $R_1 = [B_0(p_1+p_2) - B_0(p_2) - p_1^2\, C_0(p_1,p_2)]$

and $R_2 = [B_0(p_1) - B_0(p_1+p_2) - (p_2^2 + 2p_1\cdot p_2)\, C_0(p_1,p_2)]$

$$C_0(p_1,p_2) = \int [dl] \frac{1}{l^2(l+p_1)^2(l+p_1+p_2)^2}, B_0(p_1) = \int [dl] \frac{1}{l^2(l+p_1)^2}$$

Solution involves the inverse of the Gram matrix, $G_{ij} \equiv 2p_i\cdot p_j$

$$G^{-1} = \left(\begin{array}{cc} +p_2\cdot p_2 & -p_1\cdot p_2 \\ -p_1\cdot p_2 & +p_1\cdot p_1 \end{array}\right)/[2(p_1\cdot p_1\, p_2\cdot p_2 - (p_1\cdot p_2)^2)]$$

# *Historical perspective III*

- M. Veltman wrote a CDC program for numerical evaluation of the formfactors in processes with only UV divergences, Utrecht (1979).
- He dealt with exceptional regions, (e.g. regions where the Gram determinant vanishes), by implementing parts of the program in quadruple precision.
- Translation and improvement by Van Oldenborgh (1990) and further work on interface by T. Hahn and M. Perez-Victoria (1998).

However this is not sufficient for our needs.

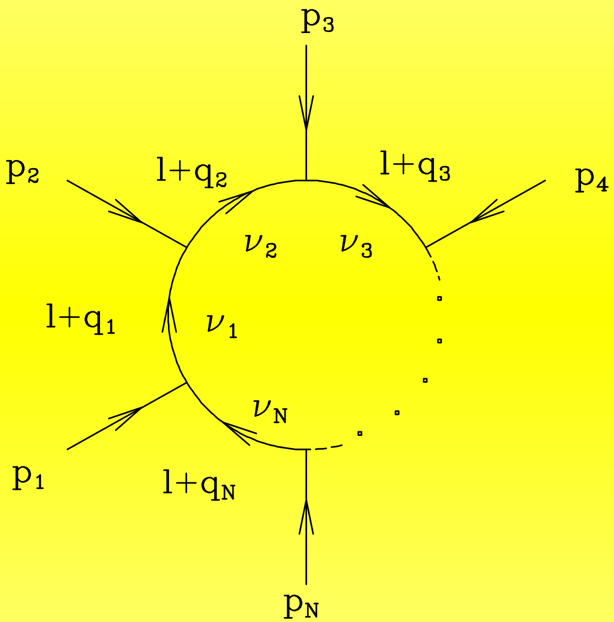
- We are interested in processes with more than 4 external legs.
- We are often interested in loop processes with collinear and soft singularities due to the presence of massless particles. These are most commonly (and elegantly) controlled by dimensional regularization.

# *Bibliography, Tensor reduction*


- D. B. Melrose, Nuovo Cimento, 1965
  In a $d$ dimensional space, a scalar diagram with $n > d$ external legs can be reduced to a sum of diagrams with $d$ external legs.
- Passarino and Veltman, Nucl. Phys. 1979
- Binoth et al., hep-ph/0504267, hep-ph/9911342
- Denner and Dittmaier, hep-ph/0509141
- Giele and Glover, hep-ph/0402152, Giele and Glover and Zanderighi hep-ph/0407016
- Anastasiou and Daleo, hep-ph/0511176

# *Recursion relations I*

Define generalized scalar integrals




$$d_i \equiv (l+q_i)^2$$

$$q_i \equiv \sum_{j=1}^{i} p_j$$

$$q_N \equiv \sum_{j=1}^{N} p_j = 0,$$

$$I(D;\nu_1,\nu_2,\ldots,\nu_N) = I(D;\{\nu_k\}_{k=1}^N) \equiv \int \frac{d^D\,l}{i\pi^{D/2}}\,\frac{1}{d_1^{\nu_1}\,d_2^{\nu_2}\cdots d_N^{\nu_N}}\,,$$

# Form-factor expansion

Davydchev

- For form factor expansion in terms of the $q$'s the coefficients are generalized scalar integrals in shifted dimensionalities
- e.g., the rank-1 and rank-2 tensor integrals with $N$ external legs can be decomposed as

$$I^{\mu_1}(D; q_1, \ldots, q_N) = \sum_{i_1=1}^{N} I(D+2; \{1+\delta_{i_1 k}\}_{k=1}^{N})\, q_{i_1}^{\mu_1}$$

$$= \quad I(D+2; 2,1,1,\ldots,1)\, q_1^{\mu_1} + I(D+2; 1,2,1,\ldots,1)\, q_2^{\mu_1}$$

$$+ \quad \cdots + I(D+2; 1,1,1,\ldots,2)\, q_N^{\mu_1}\,.$$

$$I^{\mu_1\mu_2}(D; q_1, \ldots, q_N) = -\frac{1}{2} I(D+2; 1,1,1,\ldots,1)\, g^{\mu_1\mu_2}$$

$$+2\, I(D+4; 3,1,1,\ldots,1)\, q_1^{\mu_1} q_1^{\mu_2}$$

$$+ \quad I(D+4; 2,2,1,\ldots,1)\, (q_1^{\mu_1} q_2^{\mu_2} + q_1^{\mu_2} q_2^{\mu_1}) + \cdots$$

# *Basic identity*

Tkachev,Chetyrkin,Tarasov,Duplancic,Nizic

$$\int \frac{d^D l}{i\pi^{D/2}} \frac{\partial}{\partial l^\mu} \left( \frac{\left(\sum_{i=1}^N y_i\right) l^\mu + \left(\sum_{i=1}^N y_i q_i^\mu\right)}{d_1^{\nu_1} d_2^{\nu_2} \cdots d_N^{\nu_N}} \right) = 0\,.$$

valid for arbitrary $y_i$. Differentiating we obtain the base identity

$$\sum_{j=1}^N \left( \sum_{i=1}^N S_{ji} y_i \right) \nu_j I(D; \{\nu_k + \delta_{kj}\}_{k=1}^N) = -\sum_{i=1}^N y_i I(D-2; \{\nu_k - \delta_{ki}\}_{k=1}^N)$$

$$- \quad \left( D - 1 - \sum_{j=1}^N \nu_j \right) \left( \sum_{i=1}^N y_i \right) I(D; \{\nu_k\}_{k=1}^N)\,,$$

where $S$ is a kinematic matrix which, for massless internal particles, takes the form

$$S_{ij} \equiv (q_i - q_j)^2\,.$$

# Recursion relations II

Solving $\sum_i S_{ji} y_i = \delta_{lj}$ (assuming that the inverse of the matrix $S$ exists), we derive the basic recursion relation

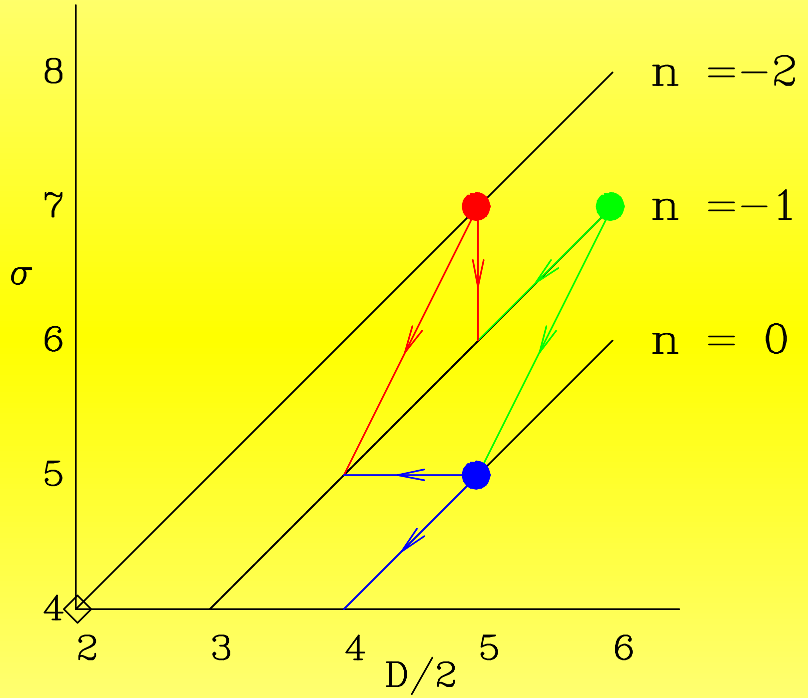
$$
\begin{aligned}
&(\nu_l - 1) I(D; \{\nu_k\}_{k=1}^N) \\
&\quad = \; -\sum_{i=1}^N S^{-1}_{li} I(D-2; \{\nu_k - \delta_{ik} - \delta_{lk}\}_{k=1}^N) \\
&\quad - \; b_l \,(D-\sigma)\, I(D; \{\nu_k - \delta_{lk}\}_{k=1}^N).
\end{aligned}
$$

$$
\sigma \equiv \sum_{i=1}^N \nu_i; \quad b_i \equiv \sum_{j=1}^N S^{-1}_{ij}; \quad B \equiv \sum_{i=1}^N b_i = \sum_{i,j=1}^N S^{-1}_{ij}\,.
$$

The strategy is to reduce more complicated integrals to a set of simpler basis integrals which are known analytically.
Hence the method is seminumerical.

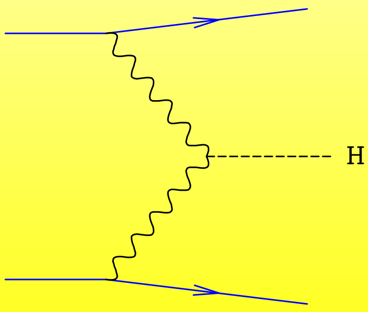
# *Recursion relations (cont)*

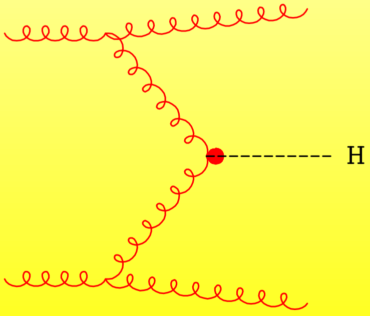
- Example: reduction of boxes, $\sigma = \sum_i \nu_i$




- Using the basic identity (red lines) and other subsidiary identities (blue and green lines) one can always arrive at the basis integral, (four-dimensional box), denoted by a diamond, (or integrals with fewer external legs).

# $H$+2 jet calculation




- NLO corrections to $W$-fusion mechanism already calculated by many authors.
- All the elements are in place for a full NLO Higgs + 2 jets calculation via gluon fusion mechanism
  - ⋆ Born level calculation Higgs + 4 partons
  - ⋆ Real calculation Higgs + 5 partons, Del Duca et al, Dixon et al, Badger et al
  - ⋆ Virtual calculation — Ellis, Giele and Zanderighi, presented here
  - ⋆ Subtraction terms — Campbell, Ellis and Zanderighi, in preparation

# *Proof of principle*

Ellis, Giele, Zanderighi

Use the effective theory ($m_t \to \infty$) for $Hgg$ coupling

$$\mathcal{L}_{\rm eff} = \frac{1}{4} A(1+\Delta) H G^a_{\mu\nu} G^{a\,\mu\nu} \,.$$

$G^a_{\mu\nu}$ is the field strength of the gluon field and $H$ is the Higgs-boson field, $A = \frac{g^2}{12\pi^2 v}$ where $g$ is the bare strong coupling and $v$ is the vacuum expectation value parameter, $v^2 = (G_F\sqrt{2})^{-1} = (246 \text{ GeV})^2$. $\Delta$ is a finite correction. Calculate virtual corrections to

$$\begin{array}{lllll} A) & H & \to & q\bar{q}q'\bar{q}', & (30 \text{ diagrams}), \\ B) & H & \to & q\bar{q}q\bar{q}, & (60 \text{ diagrams}), \\ C) & H & \to & q\bar{q}gg, & (191 \text{ diagrams}), \\ D) & H & \to & gggg, & (739 \text{ diagrams}). \end{array}$$

# *Method*

- Generate graphs and *Form* input using *Qgraf*
- Write numerical result for each diagram in terms of

$$\mathcal{A}(p_1,\ldots,p_N;\varepsilon_1,\ldots,\varepsilon_N) = \sum_{M=0}^{N} K_{\mu_1\cdots\mu_M}(p_1,\ldots,p_N;\varepsilon_1,\ldots,\varepsilon_N)$$
$$I^{\mu_1\cdots\mu_M}(D;q_1,\ldots,q_N)\,,$$

- Reduce tensor integrals numerically to a set of basis integrals (which are known analytically) by a recursive numerical procedure.
- Check the Ward identities numerically.
- Generate complete matrix elements squared by summing over squares of helicity amplitudes.

# *Comparison of numerical and analytic results for $H \to$ four partons*

| | $\frac{1}{\epsilon^2}$ | $\frac{1}{\epsilon}$ | 1 |
|---|---|---|---|
| $A_B$ | 0 | 0 | 12.9162958212387 |
| $A_{V,N}$ | -68.8869110466063 | -114.642248172519 | 120.018444115458 |
| $A_{V,A}$ | -68.8869110466064 | -114.642248172523 | 120.018444115429 |
| $B_B$ | 0 | 0 | 858.856417157052 |
| $B_{V,N}$ | -4580.56755817094 | -436.142317955208 | 26470.9608978350 |
| $B_{V,A}$ | -4580.56755817099 | -436.142317955660 | 26470.9608978346 |
| $C_B$ | 0 | 0 | 968.590160211857 |
| $C_{V,N}$ | -8394.44805516930 | -19808.0396331354 | -1287.90574949112 |
| $C_{V,A}$ | -8394.44805516942 | -19808.0396331363 | not known analytically |
| $D_B$ | 0 | 0 | 3576991.27960852 |
| $D_{V,N}$ | -4.29238953553022 $\cdot 10^7$ | -1.04436372655580 $\cdot 10^8$ | -6.79830911471604$\cdot 10^7$ |
| $D_{V,A}$ | -4.29238953553022$\cdot 10^7$ | -1.04436372655580 $\cdot 10^8$ | not known analytically |

# *Current research directions*

- NLO is the first serious approximation in QCD. We should endeavour to calculate all interesting processes at this level. MCFM represents a start in this regard, but there is much left to do.
- Stumbling block for higher leg processes: Virtual corrections There are two approaches to the evaluation of one-loop matrix elements
  - ⋆ Twistor ‘inspired’ Analytic Methods
  - ⋆ Semi-numerical or numerical methods
- We can envisage a synergy between these two approaches.
- Calculation of Matrix elements is not sufficient: Results must be cast in a form where experimenters can use them.
- Comparisons of all the Standard Model results amongst themselves and with data is crucial both for the Tevatron and the LHC.