

Geometrical Methods in Loop Calculations

A. I. Davydychev

Sugar Land / Moscow

Earlier Papers: Singularities, Reduction, etc.

L.D. Landau, Nucl. Phys. **13** (1959) 181

G. Källén and A. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, No.6 (1958) 1

S. Mandelstam, Phys. Rev. **115** (1959) 1742

R.E. Cutkosky, J. Math. Phys. **1** (1960) 429

J.C. Taylor, Phys. Rev. **117** (1960) 261

Yu.A. Simonov, Sov. Phys. JETP **16** (1963) 1599

R. Karplus, C.M. Sommerfield and E.H. Wichmann, Phys. Rev. **114** (1959) 376

A.C.T. Wu, Mat. Fys. Medd. Dan. Vid. Selsk. **33**, No.3 (1961) 1

L.M. Brown, Nuovo Cim. **22** (1961) 178

F.R. Halpern, Phys. Rev. Lett. **10** (1963) 310

B. Petersson, J. Math. Phys. **6** (1965) 1955

D.B. Melrose, Nuovo Cim. **40A** (1965) 181

G. Källén and J. Toll, J. Math. Phys. **6** (1965) 299 (1965)

B.G. Nickel, J. Math. Phys. **19** (1978) 542

N. Ortner and P. Wagner, Ann. Inst. Henri Poincaré (Phys. Théor.) **63** (1995) 81

P. Wagner, Indag. Math. **7** (1996) 527

Dimensional Regularization

One of the most powerful tools used in loop calculations is dimensional regularization.

G. 'tHooft and M. Veltman, Nucl. Phys. **B44** (1972) 189;

C.G. Bollini and J.J. Giambiagi, Nuovo Cimento **12B** (1972) 20;

J.F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289;

G.M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.

In some cases, one can derive results valid for an arbitrary space-time dimension $n = 4 - 2\epsilon$, usually in terms of various hypergeometric functions. For applications, explicit results for the terms of the expansion in ϵ are needed.

Geometrical Approach

To predict types of functions (and values of their arguments) which may appear in higher orders of ε -expansion, a geometrical approach happens to be very useful. It is summarized in the paper

A.I. Davydychev and R. Delbourgo, J. Math. Phys. **39** (1998) 4299.

Using this approach, the results for *all* terms of the ε -expansion have been obtained for the one-loop two-point function with arbitrary masses, one-loop three-point integrals with massless internal lines and arbitrary (off-shell) external momenta and two-loop vacuum diagrams with arbitrary masses.

A.I. Davydychev, Phys. Rev. **D61** (2000) 087701

These results have been represented in terms of

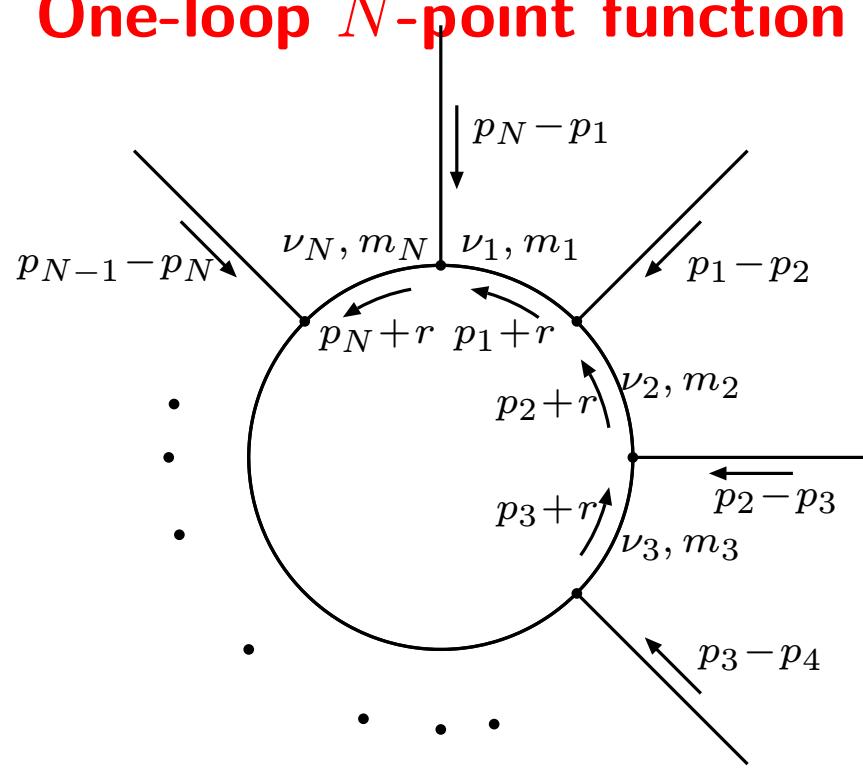
$$\text{Ls}_j(\theta) = - \int_0^\theta d\phi \ln^{j-1} \left| 2 \sin \frac{\phi}{2} \right|,$$

whose angular arguments have a rather transparent geometrical interpretation (angles of certain triangles).

In more complicated cases, generalizations of log-sine integrals (or more complex functions of angular variables) appear.

A.I. Davydychev and M.Yu. Kalmykov, Nucl. Phys. **B605** (2001) 266; Nucl. Phys. **B699** (2004) 3

One-loop N -point function



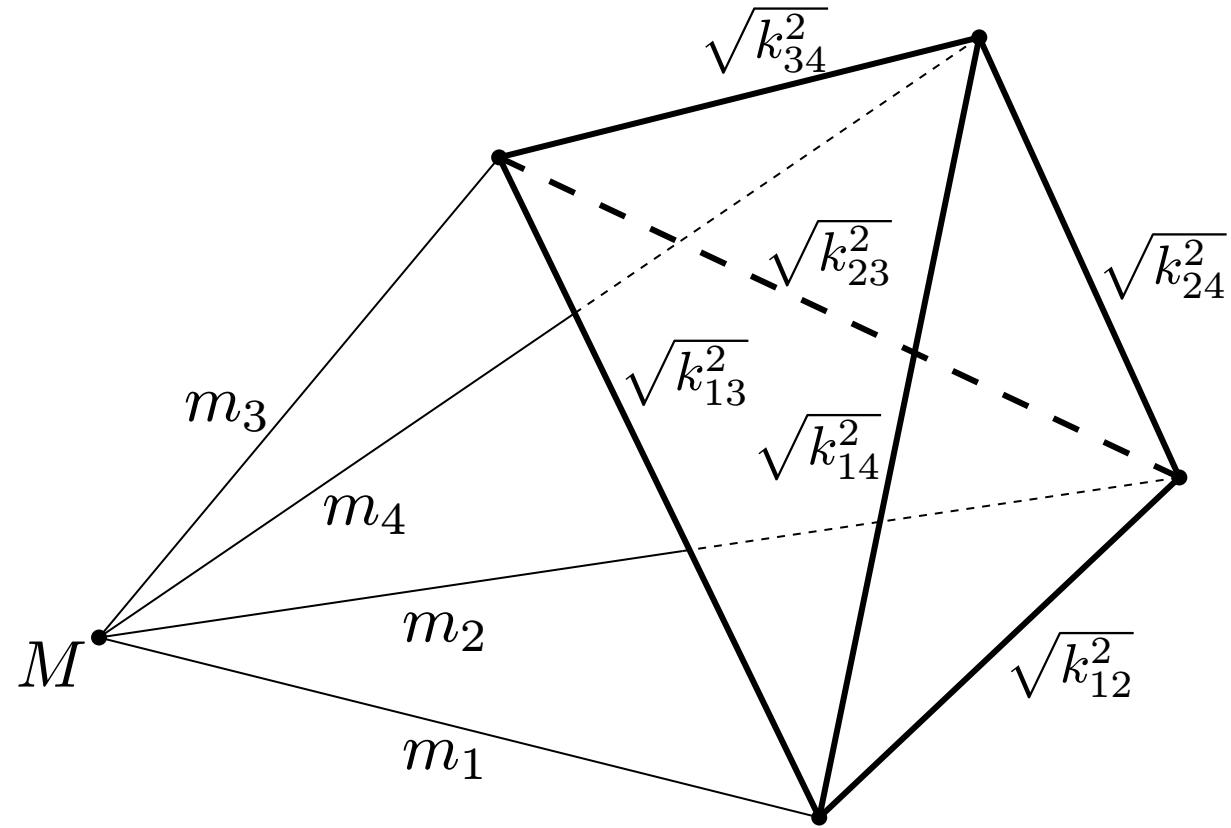
$$k_{jl}^2 = (p_j - p_l)^2$$

Feynman parameters:

$$J^{(N)}(n; \nu_1, \dots, \nu_N) = i^{1-n} \pi^{n/2} \frac{\Gamma(\sum \nu_i - n/2)}{\prod \Gamma(\nu_i)} \\ \times \int_0^1 \dots \int_0^1 \frac{\prod \alpha_i^{\nu_i-1} d\alpha_i \delta(\sum \alpha_i - 1)}{\left[\sum_{j < l} \alpha_j \alpha_l k_{jl}^2 - \sum \alpha_i m_i^2 \right]^{\sum \nu_i - n/2}}$$

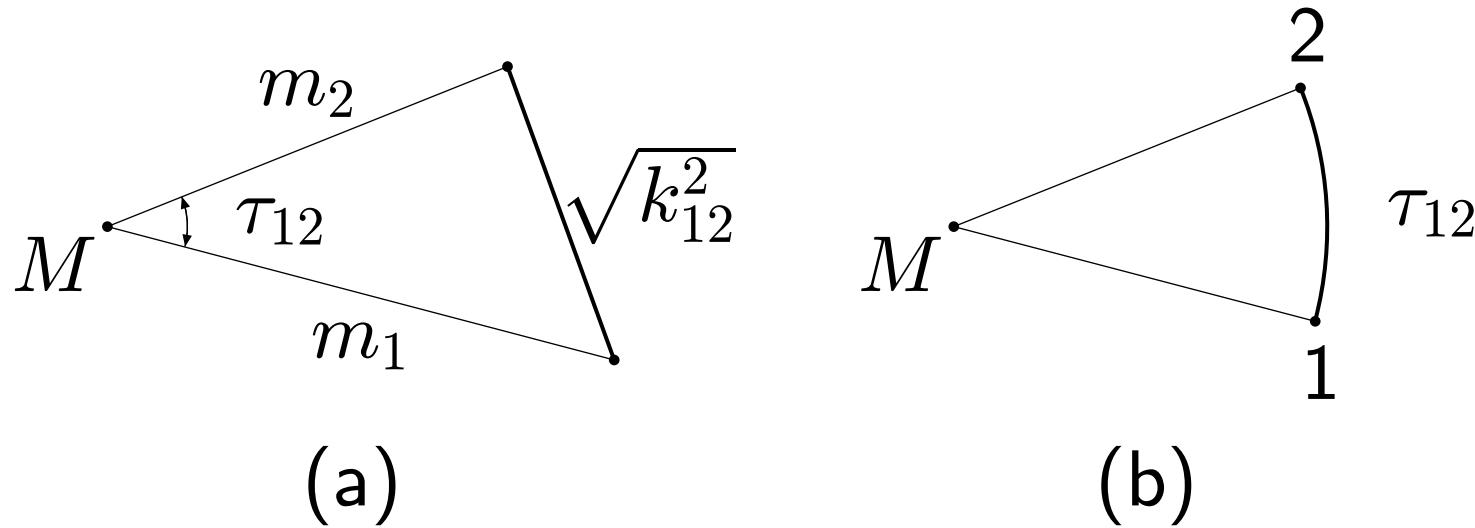
It can be transformed to an integral over the interior
of the N -dimensional solid angle $\Omega^{(N)}$ of the basic simplex,

$$J^{(N)}(n; 1, \dots, 1) = 2i^{1-2N} \pi^{n/2} \Gamma(N - \frac{n}{2}) \frac{m_0^{n-N-1}}{\sqrt{\Lambda^{(N)}}} \int_{\Omega^{(N)}} \dots \int \frac{\prod d\gamma_i}{\gamma_N^{n-N}} \delta\left(\sum \gamma_i^2 - 1\right)$$



The basic simplex for $N = 4$

General two-point function, geometrical approach



Two-point case: (a) the basic triangle and (b) the arc τ_{12}

$$\cos \tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}$$

General two-point function, geometrical approach

$$J^{(2)}(4 - 2\varepsilon; 1, 1) = i\pi^{2-\varepsilon}\Gamma(\varepsilon) \frac{m_0^{1-2\varepsilon}}{\sqrt{k_{12}^2}} \left\{ \Omega_1^{(2;4-2\varepsilon)} + \Omega_2^{(2;4-2\varepsilon)} \right\}$$

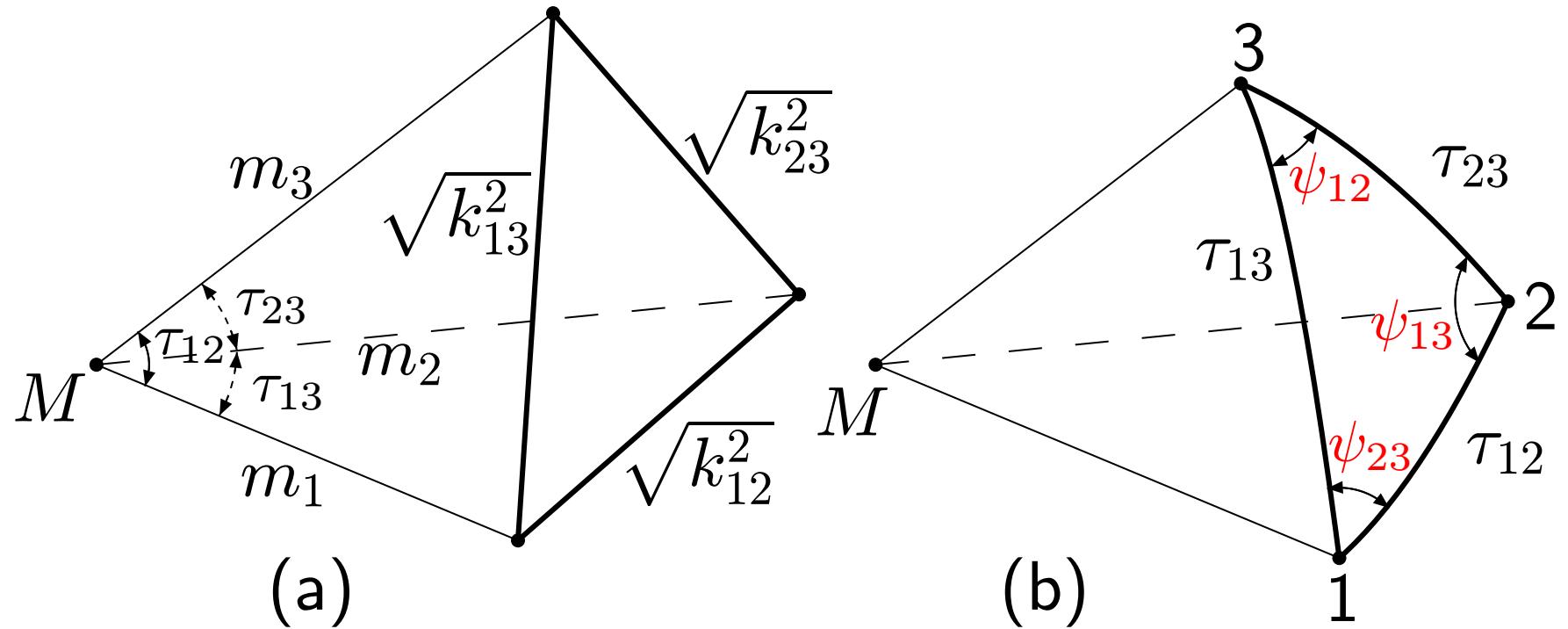
with

$$\Omega_i^{(2;4-2\varepsilon)} = \int_0^{\tau_{0i}} \frac{d\theta}{\cos^{2-2\varepsilon} \theta} = \tan \tau_{0i} {}_2F_1 \left(\begin{array}{c} 1/2, \varepsilon \\ 3/2 \end{array} \middle| -\tan^2 \tau_{0i} \right)$$

$$c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2 \tau_{12}, \quad m_0 = m_1m_2 \sqrt{\frac{D^{(2)}}{k_{12}^2}},$$

$$\cos \tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

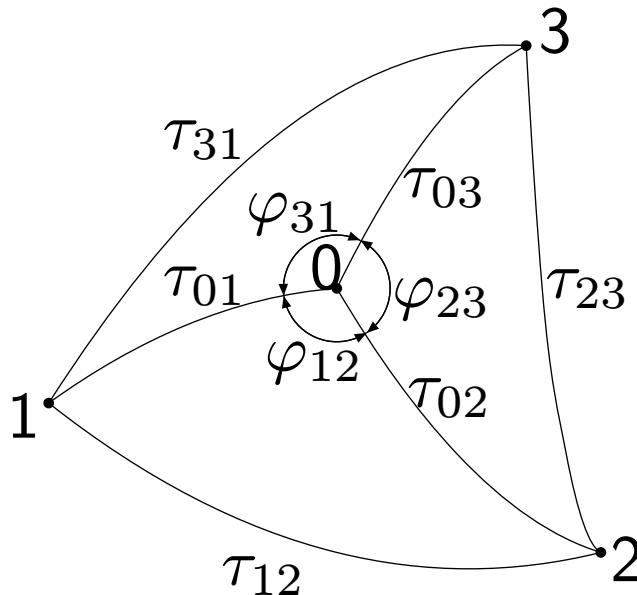
General three-point function: geometrical approach



Three-point case: (a) the basic tetrahedron and (b) the solid angle

General three-point function: geometrical approach

Relation to the angles associated with a spherical (or hyperbolic) triangle:



$$\varphi_{12} + \varphi_{23} + \varphi_{31} = 2\pi, \quad \cos \tau_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad \text{etc.}; \quad \cos \tau_{0i} = \frac{m_0}{m_i},$$

$$m_0 = m_1 m_2 m_3 \sqrt{D^{(3)} / \Lambda^{(3)}}, \quad \Lambda^{(3)} = \frac{1}{4} \Delta(k_{12}^2, k_{23}^2, k_{31}^2), \quad D^{(3)} = \det \|\cos \tau_{jl}\| \quad (j, l = 1, 2, 3)$$

Splitting into integrals over three triangles

$$J^{(3)}(n; 1, 1, 1) = \frac{m_1^2 m_2^2 m_3^2}{\Lambda^{(3)}} \sum_{i=1}^3 \frac{F_i^{(3)}}{m_i^2} J_i^{(3)}(n; 1, 1, 1)$$

where $F_i^{(3)} = \frac{\partial}{\partial m_i^2} (m_i^2 D^{(3)})$

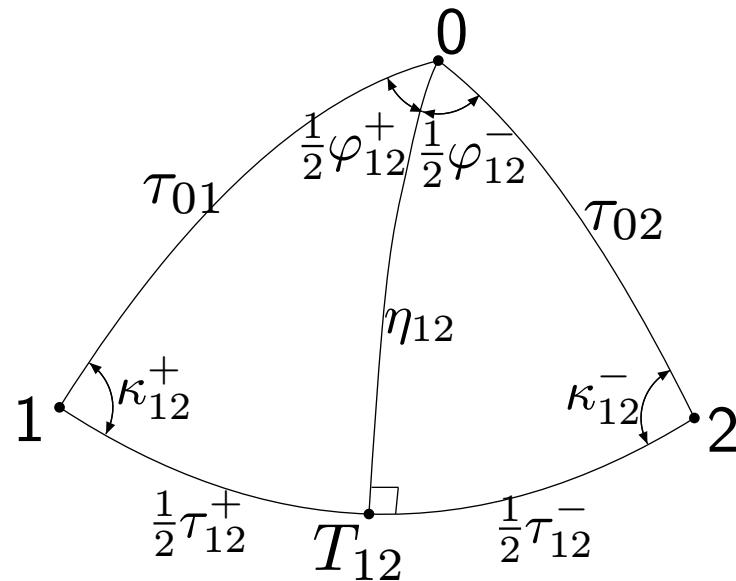
Important relation:

$$m_2^2 m_3^2 F_1^{(3)} + m_1^2 m_3^2 F_2^{(3)} + m_1^2 m_2^2 F_3^{(3)} = \Lambda^{(3)}$$

Volume of the basic tetrahedron = sum of the volumes after splitting

Further splitting into rectangular triangles

$$\frac{1}{2}(\varphi_{12}^+ + \varphi_{12}^-) = \varphi_{12}$$



Three-point function in $n = 4 - 2\varepsilon$ dimensions:

$$J^{(3)}(n; 1, 1, 1) = -\frac{i\pi^{n/2}}{\sqrt{\Lambda^{(3)}}} \Gamma\left(3 - \frac{n}{2}\right) m_0^{n-4} \Omega^{(3;n)},$$

with

$$\Omega^{(3;n)} = \int \int_{\Omega^{(3)}} \frac{\sin^{n-2} \theta \, d\theta \, d\phi}{\cos^{n-3} \theta}.$$

According to a geometrical approach,

$$\begin{aligned}\Omega^{(3;n)} = & \omega\left(\frac{1}{2}\varphi_{12}^+, \eta_{12}\right) + \omega\left(\frac{1}{2}\varphi_{12}^-, \eta_{12}\right) \\ & + \omega\left(\frac{1}{2}\varphi_{23}^+, \eta_{23}\right) + \omega\left(\frac{1}{2}\varphi_{23}^-, \eta_{23}\right) \\ & + \omega\left(\frac{1}{2}\varphi_{31}^+, \eta_{31}\right) + \omega\left(\frac{1}{2}\varphi_{31}^-, \eta_{31}\right),\end{aligned}$$

with

A.I. Davydychev, hep-th/9908032

$$\omega\left(\frac{1}{2}\varphi, \eta\right) = \frac{1}{2\varepsilon} \int_0^{\varphi/2} d\phi \left[1 - \left(1 + \frac{\tan^2 \eta}{\cos^2 \phi}\right)^{-\varepsilon} \right]$$

ε -expansion:

$$\frac{1}{2} \sum_{j=0}^{\infty} \frac{(-\varepsilon)^j}{(j+1)!} \int_0^{\varphi/2} d\phi \ln^{j+1} \left(1 + \frac{\tan^2 \eta_{12}}{\cos^2 \phi} \right)$$

In terms of Li_3 , the ε -part was calculated in

U. Nierste, D. Müller and M. Böhm, Z. Phys. **C57** (1993) 605

The result for arbitrary ε can be presented in terms of Appell's hypergeometric function F_1 ,

$$\omega\left(\frac{1}{2}\varphi, \eta\right) = \frac{1}{2\varepsilon} \left[\frac{\varphi}{2} - \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \cos^{2\varepsilon} \tau_0 F_1\left(1, 1, \varepsilon; \frac{3}{2} \middle| \sin^2 \frac{\varphi}{2}, \sin^2 \frac{\tau}{2}\right) \right],$$

with $\cos \tau_0 = \cos \eta \cos \frac{\tau}{2}$,

$$F_1(a, b, b', c|x, y) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{j+l} (b)_j (b')_l}{(c)_{j+l}} \frac{x^j y^l}{j! l!}$$

Similar functions occurred in

O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Some special cases: L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081

An important formula (shift $\varepsilon \rightarrow 1 + \varepsilon$, or $n \rightarrow n - 2$):

$$F_1(1, 1, \varepsilon; \frac{3}{2} \mid x, y) = \left(1 - \frac{y}{x}\right) F_1(1, 1, 1 + \varepsilon; \frac{3}{2} \mid x, y) + \frac{y}{x} {}_2F_1\left(\begin{array}{c} 1, 1 + \varepsilon \\ 3/2 \end{array} \middle| y\right).$$

It can be supplemented by a Kummer relation:

$$(1 - 2\varepsilon) {}_2F_1\left(\begin{array}{c} 1, \varepsilon \\ 3/2 \end{array} \middle| y\right) = 1 - 2\varepsilon(1 - y) {}_2F_1\left(\begin{array}{c} 1, 1 + \varepsilon \\ 3/2 \end{array} \middle| y\right).$$

Using these relations recursively, we can express three-point function in $n + 2j$ dimensions in terms of the three- and two-point functions in n dimensions, going from $\varepsilon - j$ to ε

Shift $n \rightarrow n + 2$: geometrical derivation

$$J^{(3)}(n+2; 1, 1, 1) = \frac{m_1^2 m_2^2 m_3^2}{\Lambda^{(3)}} \sum_{i=1}^3 \frac{F_i^{(3)}}{m_i^2} J_i^{(3)}(n+2; 1, 1, 1)$$

$$(n-2)\pi^{-1} J_3^{(3)}(n+2; 1, 1, 1) = -2m_0^2 J_3^{(3)}(n; 1, 1, 1) - J^{(3)}(n; 1, 1, 0)$$

$$\begin{aligned} J^{(3)}(n+2; 1, 1, 1) &= -\frac{\pi m_1^2 m_2^2 m_3^2}{(n-2)\Lambda^{(3)}} \left\{ 2D^{(3)} J_3^{(3)}(n; 1, 1, 1) \right. \\ &\quad \left. + \frac{F_1^{(3)}}{m_1^2} J^{(3)}(n; 0, 1, 1) + \frac{F_2^{(3)}}{m_2^2} J^{(3)}(n; 1, 0, 1) + \frac{F_3^{(3)}}{m_3^2} J^{(3)}(n; 1, 1, 0) \right\} \end{aligned}$$

Compared with O.V. Tarasov, Phys. Rev. D54 (1996) 6479

See also T. Binoth, J.Ph. Guillet, and G. Heinrich, Nucl. Phys. B572 (2000) 361;

W.T. Giele and E.W.N. Glover, JHEP 04 (2004) 029

Special values of n : $n = 3$ ($\varepsilon = \frac{1}{2}$):

We can use : $F_1(a, b, b'; b + b' | x, y) = (1 - y)^{-a} {}_2F_1\left(\begin{array}{c} a, b \\ b + b' \end{array} \middle| \frac{x - y}{1 - y}\right)$,

$${}_2F_1\left(\begin{array}{c} 1, 1 \\ 3/2 \end{array} \middle| z\right) = \frac{\arcsin \sqrt{z}}{\sqrt{z(1 - z)}},$$

We get $\omega\left(\frac{1}{2}\varphi, \eta\right) \Big|_{n=3} = \frac{\varphi}{2} - \frac{\pi}{2} + \kappa$, with $\cos \kappa = \sin \frac{\varphi}{2} \cos \eta$.

Collect results for all six triangles:

$$\Omega^{(3;3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi.$$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Special values of n : $n = 2$ ($\varepsilon = 1$):

$$F_1 \left(1, 1, 1; \frac{3}{2} \mid x, y \right) = \frac{1}{x - y} \left[\frac{\sqrt{x} \arcsin \sqrt{x}}{\sqrt{1-x}} - \frac{\sqrt{y} \arcsin \sqrt{y}}{\sqrt{1-y}} \right] .$$

In this way, we get

$$\omega \left(\frac{1}{2}\varphi, \eta \right) \Big|_{n=2} = \frac{\tau}{4} \sin \eta .$$

Collecting results for all six triangles, we get

$$\Omega^{(3;2)} = \frac{1}{2} (\tau_{12} \sin \eta_{12} + \tau_{23} \sin \eta_{23} + \tau_{13} \sin \eta_{13}) .$$

The two-point integral in two dimensions is proportional to $\tau / \sin \tau$
 \Rightarrow three-point integral in two dimensions is a combination of three two-point integrals, with coefficients proportional to $\sin \tau_{jl} \sin \eta_{jl}$.

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Special values of n : $n = 5$ ($\varepsilon = -\frac{1}{2}$):

In this case, we obtain

$$\omega \left(\frac{1}{2}\varphi, \eta \right) \Big|_{n=5} = -\frac{\varphi}{2} + \frac{\pi}{2} - \kappa + \frac{1}{2} \tan \eta \ln \left(\frac{1 + \sin \frac{\tau}{2}}{1 - \sin \frac{\tau}{2}} \right).$$

Collecting results for all six triangles, we obtain

$$\begin{aligned} \Omega^{(3;5)} = & -(\psi_{12} + \psi_{23} + \psi_{31} - \pi) + \tan \eta_{12} \ln \left(\frac{m_1 + m_2 + \sqrt{k_{12}^2}}{m_1 + m_2 - \sqrt{k_{12}^2}} \right) \\ & + \tan \eta_{23} \ln \left(\frac{m_2 + m_3 + \sqrt{k_{23}^2}}{m_2 + m_3 - \sqrt{k_{23}^2}} \right) + \tan \eta_{13} \ln \left(\frac{m_1 + m_3 + \sqrt{k_{13}^2}}{m_1 + m_3 - \sqrt{k_{13}^2}} \right). \end{aligned}$$

In other words, the five-dimensional three-point integral can be expressed in terms of the three-dimensional three- and two-point integrals.

Special values of n : $n = 4$ ($\varepsilon \rightarrow 0$):

$$\int_0^{\varphi/2} d\phi \ln \left(1 + \frac{\tan^2 \eta}{\cos^2 \phi} \right) = \frac{1}{2}\tau \ln \left(\frac{1 + \sin \eta}{1 - \sin \eta} \right) + \frac{1}{2} \text{Cl}_2(\varphi + \tau) + \frac{1}{2} \text{Cl}_2(\varphi - \tau) - \text{Cl}_2(\varphi)$$

Compare with: P. Wagner, Indag. Math. 7 (1996) 527

After analytical continuation, corresponds to

G. 'tHooft and M. Veltman, Nucl. Phys. B153 (1979) 365

Analytic Continuation: Arbitrary Dimension

Consider $\int_0^{\varphi_0} d\phi \left(1 + \frac{\tan^2 \eta}{\cos^2 \phi}\right)^{-\varepsilon}$.

Substitute $z \Rightarrow e^{2i\phi}$, so that $\cos^2 \phi \Rightarrow \frac{(1+z)^2}{4z}$,

$$1 + \frac{\tan^2 \eta}{\cos^2 \phi} \Rightarrow \frac{(z+\rho)(z+1/\rho)}{(z+1)^2}, \quad \text{with } \rho \equiv \frac{1-\sin \eta}{1+\sin \eta}$$

In this way, $\int_0^{\varphi_0} d\phi \left(1 + \frac{\tan^2 \eta}{\cos^2 \phi}\right)^{-\varepsilon} \Rightarrow \frac{i}{2} \int_{z_0}^1 \frac{dz}{z} \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right]^{-\varepsilon}$,

with $z_0 \leftrightarrow e^{2i\varphi_0}$.

Analytic Continuation: Expansion in ε

Expanding in ε , we get

$$Q_j \equiv \int_{z_0}^1 \frac{dz}{z} \ln^j \left[\frac{(z + \rho)(z + 1/\rho)}{(z + 1)^2} \right] .$$

The first term, $\mathcal{O}(1)$:

$$\begin{aligned} Q_1 &\equiv \int_{z_0}^1 \frac{dz}{z} \ln \left[\frac{(z + \rho)(z + 1/\rho)}{(z + 1)^2} \right] \\ &= \text{Li}_2(-z_0\rho) + \text{Li}_2(-z_0/\rho) - 2\text{Li}_2(-z_0) + \frac{1}{2}\ln^2\rho \end{aligned}$$

Analytic Continuation: Expansion in ε (continued)

$$\begin{aligned}
 Q_2 &\equiv \int_{z_0}^1 \frac{dz}{z} \ln^2 \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \\
 &= \ln \rho \left[2\text{Li}_2 \left(\frac{1-\rho}{1+z_0\rho} \right) + 2\text{Li}_2 \left(\frac{z_0(\rho-1)}{1+z_0\rho} \right) - 2\text{Li}_2 \left(\frac{\rho-1}{z_0+\rho} \right) - 2\text{Li}_2 \left(\frac{z_0(1-\rho)}{z_0+\rho} \right) \right. \\
 &\quad \left. - \text{Li}_2 \left(\frac{1-\rho^2}{1+z_0\rho} \right) - \text{Li}_2 \left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)} \right) + \text{Li}_2 \left(\frac{\rho^2-1}{\rho(z_0+\rho)} \right) + \text{Li}_2 \left(\frac{z_0(1-\rho^2)}{z_0+\rho} \right) \right] \\
 &\quad + 4S_{1,2} \left(\frac{1-\rho}{1+z_0\rho} \right) - 4S_{1,2} \left(\frac{z_0(\rho-1)}{1+z_0\rho} \right) + 4S_{1,2} \left(\frac{\rho-1}{z_0+\rho} \right) - 4S_{1,2} \left(\frac{z_0(1-\rho)}{z_0+\rho} \right) \\
 &\quad - S_{1,2} \left(\frac{1-\rho^2}{1+z_0\rho} \right) + S_{1,2} \left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)} \right) - S_{1,2} \left(\frac{\rho^2-1}{\rho(z_0+\rho)} \right) + S_{1,2} \left(\frac{z_0(1-\rho^2)}{z_0+\rho} \right)
 \end{aligned}$$

Compare with: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

ε -expansion: higher terms

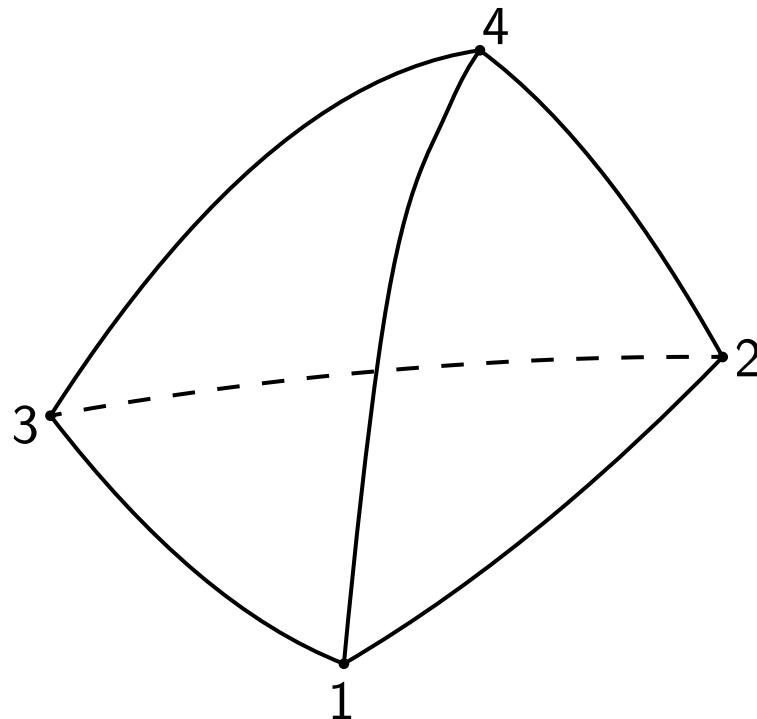
ε^2 -term

$$Q_3 = \int_{z_0}^1 \frac{dz}{z} \ln^3 \left[\frac{(z + \rho)(z + 1/\rho)}{(z + 1)^2} \right],$$

etc.

Some special cases considered in J.G. Körner, Z. Merebashvili, M. Rogal, Phys. Rev. D71 (2005) 054028

Geometrical approach: 4-point function



The spherical tetrahedon

