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HOW MANY CONSERVED CURRENTS ARE NECESSARY FOR THE CALCULATION OF THE S-MATRIX IN THE MASSIVE THIRRING MODEL?

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ABSTRACT

In the quantized massive Thirring model, I show that for repulsive coupling, the existence of one, conserved, local, rank-four Lorentz tensor is a sufficient condition for the absence of particle production and the factorization of the S-matrix.

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Recently Zamolodchikov,[1] Karowski and Thun,[2] and Karowski, Thun, Truong and Weisz[3] calculated the exact on-mass shell S-matrix for the quantized massive Thirring model or equivalently the quantum sine-Gordon model.[4] Their calculations assumed both the absence of particle production and the factorization of the N-body S-matrix into a sum of products of two-body S-matrices. These properties are known to be implied by the existence, in the quantized model, of an infinite number of conserved local currents which transform according to higher and higher rank representations of the Lorentz group.[5]

The existence of such currents has been known for some time in the classical sine-Gordon model.[6] Once their importance to the quantum theory was realized, it was quickly shown that the quantum perturbation theory about the free sine-Gordon model produces no anomalies for some of the lower-rank tensor currents.[7] Similar results were obtained for both the classical[8] and quantized[9] massive Thirring model shortly thereafter. However, only last year, Lowenstein and Speer[10] showed that all tensor currents survive quantization in the quantum perturbation expansion about both the free sine-Gordon model and the free massive Thirring model.

In this note, I show that for the region of coupling constant space for which there are no bosons in the particle spectrum (referred to as the repulsive coupling region) the existence of one, conserved, local, rank-four Lorentz tensor is a sufficient condition for the absence of particle production and the factorization of the S-matrix. Therefore, for repulsive coupling, an infinite number of tensor currents in the quantized model is not logically necessary for the calculation of the exact S-matrix.

The proof rests on two assumptions about the model: the content of the particle spectrum and the existence of a conserved tensor. The particle

spectrum consists of a fermion, antifermion, and a finite number of bosons. These bosons are bound states of a fermion and an antifermion. While their number depends on the value of the coupling constant, there is a region of coupling constant space for which there are no bosons.

The second assumption is the existence of a conserved, rank-four tensor, $T^{\mu\nu\lambda\delta}$, in the model. This tensor has the following properties: it is a local polynomial in the fields and its derivatives; it is conserved with respect to the first index; it is symmetric and traceless with respect to the last three indices; and it transforms like a rank-four representation of the Lorentz group. These properties of $T^{\mu\nu\lambda\delta}$ have been shown to survive quantization to all orders in both perturbation expansions of the model.[7,9] The existence of such a tensor has strong implications for the theory regardless of whether or not there are bound states in the particle spectrum.

The conserved charge derived from this current, $Q^{\nu\lambda\delta} \equiv \int dx\, T^{0\nu\lambda\delta}$, has only two non-zero components Q^{+++} and Q^{---}, where $\pm$ refers to the light-cone coordinates $x^{\pm} = x^0 \pm x^1$. These components of $Q^{\nu\lambda\delta}$ commute with each other, the fermionic charge operator, the momentum operator and hence the mass operator. Thus, the single particle momentum eigenstates are eigenstates of $Q^{\nu\lambda\delta}$. The eigenvalue is dictated by Lorentz invariance. Therefore,

$$Q^{+++}|p_a\rangle = \eta_a (p_a^+)^3 |p_a\rangle$$

and a similar equation for Q^{---}. $|p_a\rangle$ is a momentum eigenstate for the a^{th} particle with eigenvalue p_a^μ. The Lorentz scalar, η_a, gives asymptotic conservation of $Q^{\nu\lambda\delta}$ and in general depends on the particle type. However, η must be the same for the fermion and antifermion.

Since $Q^{\nu\lambda\delta}$ is the integral of a local current density, the action on widely separated, multi-particle states is the sum of the actions on the individual particles. Therefore,

$$Q^{+++}|p_1 \ldots p_n\rangle = \left[\sum_{i=1}^{n} \eta_i (p_i^+)^3\right] |p_1 \ldots p_n\rangle$$

and similarly for Q^{---}. Thus in a scattering process, as well as the asymptotic conservation of $\sum p_i^+$ and $\sum p_i^-$, the model also has the asymptotic conservation of $\sum \eta_i (p_i^+)^3$ and $\sum \eta_i (p_i^-)^3$.

To proceed further, I restrict myself to the region of coupling constant space for which the fermion and the antifermion are the only particles in the theory. For this region of coupling constant space, I show that two particles can only scatter into two particles, and that N particles scatter effectively two at a time. From this is deduced the absence of particle production and the factorization of the S-matrix.

The conventions throughout this note are such that the mass of the fermion is equal to one, η for the fermion is equal to one, and the word particle is used to mean fermion and/or antifermion.

In proving that two particles can only scatter into two particles,[11] I use the asymptotic conservation of the four quantities determined earlier. In the general scattering of two particles into N particles, the incoming and outgoing momenta are labeled q_i^μ and p_i^μ, respectively. Since all momenta are on the mass shell, $q_i^- = (q_i^+)^{-1}$ and $p_i^- = (p_i^+)^{-1}$. I work in the zero momentum frame; thus, $q_1^+ = q_2^- \equiv \alpha$ and $q_1^- = q_2^+ = \alpha^{-1}$. Also, p_i^+ is denoted by x_i. The conservation laws then imply that:

$$\alpha + \frac{1}{\alpha} = \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} \frac{1}{x_i} ,$$

and

$$\alpha^3 + \frac{1}{\alpha^3} = \sum_{i=1}^{N} x_i^3 = \sum_{i=1}^{N} \frac{1}{x_i^3} .$$

Using the identities $(\alpha + 1/\alpha)^3 - (\alpha^3 + 1/\alpha^3) - 3(\alpha + 1/\alpha) = 0$, and $(\sum_i x_i)^3 = \sum_i x_i^3 + 2(\sum_{i>j} x_i x_j)(\sum_k x_k) + \sum_{i\neq j} x_i x_j^2$, I obtain the following:

$$2\left(\sum_{i>j} (x_i x_j)^\sigma - \frac{3}{2}\right)\left(\sum_k x_k^\sigma\right) + \sum_{i\neq j} (x_i x_j^2)^\sigma = 0 \qquad \text{with } \sigma = \pm 1 .$$

Because all the x_i's are positive numbers, the following inequality must hold

$$\sum_{i>j} (x_i x_j)^\sigma < \frac{3}{2} \qquad \text{with } \sigma = \pm 1.$$

Then, by ordering the x_i's such that $x_1 \le x_2 \le \dots \le x_N$, these inequalities can be combined, giving

$$\frac{N(N-1)}{3} < x_N x_{N-1} < \frac{3}{2} .$$

But for $N > 2$, $N(N-1)/3 > 3/2$. Thus N must be equal to 2, since $N = 1$ is forbidden by energy and momentum conservation alone. Hence if two particles collide, independently of any other particles, the out-state is the same as the in-state except for a possible exchange of fermion number.

Before I can show that N particles scatter effectively two at a time, the effect of $Q^{\nu\lambda\delta}$ on a localized wavepacket[12] must be understood. However, since all particles have the same mass and η, it is more convenient to consider the effect of $Q_p \equiv [Q^{+++} - Q^{---} - 3(P^+ - P^-)]/8$ on such a wavepacket, where P^μ is the momentum operator. A one-particle momentum eigenstate, with space-momentum eigenvalue p, is also an eigenstate of Q_p with eigenvalue p^3.

The wavefunction used to represent a single-particle state with mean momentum $\bar{p}$ is

$$\psi(x,t) = N\int_{-\infty}^{+\infty} dp\, e^{-(p-\bar{p})^2/2(\bar{E}\delta\phi)^2}\, e^{i(p(x-x_0)-E(t-t_0))} \qquad (1)$$

where N is a normalization factor, $\bar{E} = \sqrt{\bar{p}^2+1} \equiv \bar{\gamma}$ and $0 < \delta\phi << 1$. The physical interpretation of $\delta\phi$ is that it is half the velocity spread of the wavepacket when viewed from the frame in which $\bar{p} = 0$. The condition $\delta\phi << 1$ insures that the minimum spread is much larger than the Compton wavelength in this frame and thus enables me to avoid the problems associated with localizing a relativistic particle in too small a region.

At any particular time, the region of space where the particle is most likely to be found is also of interest. This region is found by performing a stationary phase analysis on the integral in Eq. (1). This analysis shows that the center of the wavepacket at time, t, is $\bar{x}(t)$ given by

$$\bar{x}(t) = x_0 + \bar{v}(t-t_0)$$

where $\bar{v} \equiv \bar{p}/\bar{E}$. The extent of the wavepacket about $\bar{x}(t)$ is defined as $\delta x(t)$ where

$$\delta x(t) = \kappa|t-t_0|(\delta\phi/\bar{\gamma}^2)$$

for large $|t-t_0|$. For small $|t-t_0|$, $\delta x(t)$ is a constant, whose value will be of no concern. $\delta\phi/\bar{\gamma}^2$ is half the velocity spread of the wavepacket as measured in the frame in which the mean velocity is $\bar{v}$. The quantity κ is chosen such that the probability of finding the particle outside the interval $(\bar{x}-\delta x, \bar{x}+\delta x)$ is extremely small. There is no point in discussing in detail the criteria for choosing κ; as will become clear later in this note, the arguments are valid for any non-zero κ.

The action of $\exp[ibQ_p/3]$ on such a wavepacket is to replace the wavefunction $\psi(x,t)$ with

$$\tilde{\psi}(x,t) = N\int_{-\infty}^{+\infty} dp\, e^{-(p-\bar{p})^2/2(\bar{E}\delta\phi)^2}\, e^{i\left(p(x-x_0)-E(t-t_0)\right)}\, e^{ibp^3/3} .$$

Again, using stationary phase analysis, the center of the wavepacket at time t is given by

$$\bar{x}(t) = (x_0 - b\bar{p}^2) + \bar{v}(t-t_0) .$$

Also the extent of the wavepacket about $\bar{x}(t)$ is $\delta x(t)$ where

$$\delta x(t) = \kappa|t-(t_0 + 2b\bar{p}\bar{E}\bar{\gamma}^2)|(\delta\phi/\bar{\gamma}^2)$$

for large $|t-(t_0 + 2b\bar{p}\bar{E}\bar{\gamma}^2)|$.

From these last two equations, note that the transformation generated by Q_p moves the center of the wavepacket and the time of minimum spread by an amount that depends on the mean momentum.

A general N particle scattering process can be represented by the impinging of N such wavepackets. The i^{th} wavepacket has mean momentum $\bar{p}_i$ and has minimum spread at time t_0^i at which time the center is at x_0^i. However, to identify N-independent wavepackets in the far past, I require that the asymptotic extent of no two wavepackets can overlap. This is accomplished by restricting the momenta such that

$$|\bar{v}_i - \bar{v}_j| > \kappa(\bar{\gamma}_i^{-2} + \bar{\gamma}_j^{-2})\delta\phi \qquad (2)$$

for all pairs $i \neq j$.

For the moment, consider N non-interacting particles in such a scattering configuration. The volume of space-time where two or more wavepackets overlap is the possible region of interaction. Therefore the interaction region for the i^{th} and j^{th} wavepacket is defined as that region of space-time for which the extent of the i^{th} wavepacket overlaps the extent of the j^{th} wavepacket.

After applying a large transformation generated by Q_p, to this N particle process, the coordinates of the center of the interaction region for the i^{th} and j^{th} particle, t_{ij} and r_{ij}, are given by

$$t_{ij} = b\,\bar{E}_i\bar{E}_j(\bar{p}_i^2 - \bar{p}_j^2)/(\bar{p}_i\bar{E}_j - \bar{p}_j\bar{E}_i) + O(b^0)$$

and

$$r_{ij} = b\,\bar{p}_i\bar{p}_j(\bar{p}_i\bar{E}_i - \bar{p}_j\bar{E}_j)/(\bar{p}_i\bar{E}_j - \bar{p}_j\bar{E}_i) + O(b^0) .$$

The terms $O(b^0)$ contains the t_0's and x_0's, but for large b these terms are of no interest.

If the particles are described in terms of their rapidities, that is $\bar{p}_i$ = $\sinh\phi_i$ and $\bar{E}_i = \cosh\phi_i$, then:

$$t_{ij} = b \cosh\phi_i \cosh\phi_j \sinh(\phi_i + \phi_j) + O(b^0) \tag{3a}$$

and

$$r_{ij} = b \sinh\phi_i \sinh\phi_j \cosh(\phi_i + \phi_j) + O(b^0) \ . \tag{3b}$$

These expressions are important because, for fixed large b, given t_{ij} and r_{ij}, they uniquely specify ϕ_i and ϕ_j. This statement is obvious for $|\phi_i|$ and $|\phi_j|$ much less than 1, since $t_{ij} \approx b(\phi_i + \phi_j)$ and $r_{ij} \approx b\,\phi_i\,\phi_j$. But it can also be shown for arbitrary rapidities. Thus, the coordinates of the center of a two-particle interaction region, after a large transformation generated by Q_p, uniquely specifies the momentum of both particles.

To determine whether these two-particle interaction regions can overlap, their extent in space-time should be calculated. This can be done by replacing the centers of the wavepackets by the extent of the wavepackets about their centers, when calculating t_{ij} and r_{ij}. That is, $\bar{x}(t) \pm \delta x(t)$ is used instead of $\bar{x}(t)$ for both particles. However, since $\delta x(t)$ was determined by replacing $\bar{p}$ by $\bar{p} \pm \kappa \bar{E} \delta\phi$, it is easier to vary the rapidities directly. Thus, by replacing ϕ_i by $\phi_i \pm \kappa\delta\phi$ and ϕ_j by $\phi_j \pm \kappa\delta\phi$ in Eq. (3), the region of space-time for which these two wavepackets overlap can be found.

However, this is not necessary for determining whether the interaction regions overlap, because of the restriction imposed on the possible rapidities. This restriction, Eq. (2), comes from the need to be able to distinguish all the particles in the far past. When rewritten in terms of the rapidities, it is $|\phi_i - \phi_j| > 2\kappa\delta\phi$ for all pairs $i \neq j$. Hence, there are no particles with rapidities such that two or more two-particle interaction regions can overlap. Thus, after a large transformation generated by Q_p, the possible interaction regions for an N particle scattering process are separate, distinct two-particle interaction regions.

If interactions are now allowed, the above situation does not change. I still have the freedom to make transformations generated by Q_p, and I

have already shown that two particles can only scatter into two particles. Thus, there is no particle production in an N particle collision. Also, because the interaction region of any two particles can be made arbitrarily far from any other interaction region, by taking the parameter b to infinity, the N particle S-matrix factorizes into a product of two-particle S-matrices. As a consequence of these results the outgoing momentum set is equal to the incoming momentum set.

In concluding, I have shown that for the region of coupling constant space for which there are no bosons in the quantized massive Thirring model, the S-matrix can be calculated assuming the existence of one, conserved, local, rank-four Lorentz tensor. The extension of this result to the region of coupling constant space for which there are bosons in the particle spectrum has also been considered. However, at the time of writing, a general proof of the absence of particle production has yet to be completed.

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