Fermi National Accelerator Laboratory

FERMILAB-Pub-85/31-T
March, 1985

# PERTURBATIVE QCD UTILIZING EXTENDED SUPERSYMMETRY

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## ABSTRACT

We use N=2 extended supersymmetry to dramatically simplify the diagrammatic calculations of perturbative QCD. The production, by gluon-gluon fusion, of two gluons, three gluons, and two massless gluinos plus a gluon are used as examples.

Standard applications of perturbative QCD [1] for hadrons involve the convolution of parton scattering cross sections with hadron structure functions. The calculation of these parton scattering cross sections by the use of Feynman diagrams is the subject of this paper. It is well known that the evaluation of QCD Feynman diagrams rapidly becomes extremely complicated, even at tree level, as the order of perturbation theory increases. This complexity arises because of the difficulty of performing algebraic manipulations with a large number of terms. Consider multi-jet production at hadron colliders. Two-jet production is easy to handle; the most complicated subset of diagrams corresponds to the elastic scattering of two gluons, which can be easily calculated by hand. Considerable complications appear in three-jet production because of the contribution from two gluons scattering into three gluons. There are twenty five diagrams for this purely gluonic process, all of them containing at least one three-gluon vertex. Thus, the cross section is virtually impossible to compute by hand; symbolic manipulation programs make it barely tractable.

In this letter we present a new technique for calculating QCD Feynman diagrams, which enables dramatic simplification of such computations. The main idea is to embed QCD in a minimal N=2 supersymmetric extension such that to tree level the two theories are identical for quarks and gluons. In this extended theory there are simple relationships between vector ( gluon ) scattering amplitudes and scalar scattering amplitudes when expressed in terms of the helicities of the external particles. Loosely speaking, supersymmetry allows the replacement of some external gluon lines by scalar lines. Thus a vector scattering amplitude is calculated by first calculating the appropriate scalar amplitude. This procedure causes a great reduction in the number of terms involved and hence great simplifications. First, most, if not all, the troublesome three-gluon vertices are absent and second, most of problems associated with the polarization vectors for the external gluons are circumvented. Cross sections are obtained by summing the squares of the different helicity amplitudes.

More precisely, we embed QCD in its minimal supersymmetric extension which contains the scalar particles in the gauge hypermultiplet. This is the SO(2) extended supersymmetric QCD [2]. The SO(2) model describes gluons, two species of gluinos, one complex gauge scalar in the adjoint representation, quarks, squarks, mirror quarks and mirror squarks. In terms of N=1 superfields [3], the SO(2) gauge hypermultiplet consists of one gauge vector superfield V ( gluon g and gluino $\lambda$ ) and one chiral superfield X ( gluino $\chi$ and complex scalar $\phi$ ) in the adjoint representation. The matter hypermultiplet consists of left-handed quark superfield L ( left-handed quark q and squark $\sigma$ ) and right-handed mirror superfield $R^\dagger$ ( right-handed mirror quark $\bar{r}$ and mirror squark $\rho^\dagger$ ). The SO(2) Lagrangian [2], written in terms of N=1 superfields, is

$$\mathscr{L} = \frac{1}{8g^2}[WW]_F + [\sqrt{2}igRXL]_F + \text{h.c.}$$

$$+ [2\mathrm{Tr}X^\dagger e^{2gV} X e^{-2gV} + L^\dagger e^{2gV} L + R^\dagger e^{-2gV^T} R]_D. \qquad (1)$$

Let $Q^a_\alpha$, a=1,2, be the supersymmetry generators which transform like Majorana spinors and commute with the S-matrix. From these, we define two hermitian operators, $Q^a(\eta^a) = \bar{\eta}^a_\alpha Q^a_\alpha$, where $\eta$ is an arbitrary Majorana spinor, parametrized as follows:

$$\eta = \tfrac{1}{2} [\, \eta_1 + \eta_2^* \,,\, -\eta_1^* + \eta_2 \,,\, -\eta_1 + \eta_2^* \,,\, -\eta_1^* - \eta_2 \,]. \tag{2}$$

Unbroken extended supersymmetry implies that the operators $Q^a(\eta)$ annihilate the vacuum: $Q^a(\eta)|vac\rangle = 0$. We are interested in scattering amplitudes, i.e. the vacuum expectation values of the products of the creation and annihilation operators. Therefore, let us denote by $z^i_s(p_i)$ the annihilation operator for the particle $z(=g,\lambda,\chi,\phi,q,\sigma,\bar{r},\rho^\dagger)$ of helicity $s(=+,-)$ and momentum

$$p_i = E_i \,(\, 1 \,,\, \sin\theta_i \cos\beta_i \,,\, \sin\theta_i \sin\beta_i \,,\, \cos\theta_i \,). \tag{3}$$

The supersymmetry operators $Q^a(\eta)$ act on these annihilation operators in the following way [4]:

$$[Q^a(\eta),\lambda^b_\pm(p)] = \mp\Gamma^\mp(p,\eta)g_\pm\delta^{ab} \mp i\Gamma^\pm(p,\eta)\phi_\pm\varepsilon^{ab} \tag{4}$$

$$[Q^a(\eta),g_\pm(p)] = \mp\Gamma^\pm(p,\eta)\lambda^a_\pm \tag{5}$$

$$[Q^a(\eta),\phi_\pm(p)] = \pm i\Gamma^\mp(p,\eta)\varepsilon^{ab}\lambda^b_\pm \tag{6}$$

$$[Q^a(\eta),\zeta^b_\pm(p)] = \pm i\Gamma^\mp(p,\eta)q_\pm\delta^{ab} \mp i\Gamma^\mp(p,\eta)\bar{r}_\pm\varepsilon^{ab} \tag{7}$$

$$[Q^a(\eta),\bar{r}_\pm(p)] = \pm i\Gamma^\pm(p,\eta)\varepsilon^{ab}\zeta^b_\pm \tag{8}$$

$$[Q^a(\eta),q_\pm(p)] = \mp i\Gamma^\pm(p,\eta)\zeta^a_\pm \,, \tag{9}$$

where

$$\Gamma^+(p,\eta) = [\Gamma^-(p,\eta)]^* = \sqrt{2E}[\eta_1\cos\theta/2e^{i\beta/2} + \eta_2\sin\theta/2e^{-i\beta/2}] \tag{10}$$

and the conventions of Refs[4] are used.

In Eqs(4)-(9), the notation $\lambda^1=\lambda$, $\lambda^2=\chi$, $\zeta^1=\sigma$, $\zeta^2=\rho^\dagger$, is used in order to make the SO(2) symmetry manifest. We adopted the convention that for the scalar particles the helicity plus annihilation operator annihilates the particle, whereas the helicity minus operator annihilates its complex conjugate. Since supersymmetry commute with the S-matrix, both "in" and "out" operators transform in the same way. The commutators for the creation operators are obtained from Eqs(4)-(9) by hermitian conjugation. Care must be taken with signs when commuting $\Gamma(p,\eta)$ as $\Gamma$ is a Grassmann variable.

The relations between different scattering amplitudes can be obtained in the following way [4]. Since $Q^a(\eta)|vac\rangle = 0$, we have:

$$0 = \langle vac|[Q^a, z^1_{out}\cdots z^m_{out} z^{1\dagger}_{in}\cdots z^{n\dagger}_{in}]|vac\rangle$$

$$= \sum_i \langle vac|z^1_{out}\cdots[Q^a, z^i_{out}]\cdots|vac\rangle$$

$$+ \sum_j \langle vac|z^1_{out}\cdots[Q^a, z^{j\dagger}_{in}]\cdots|vac\rangle. \tag{11}$$

Inserting the commutators, Eqs(4)-(9), we derive linear relations between amplitudes, with coefficients that are either $\Gamma^+(\eta)$ or $\Gamma^-(\eta)$. These relations split into four separate equations, since $\eta_1$, $\eta_2$, $\eta_1^*$, $\eta_2^*$ are independent. They relate amplitudes for particles with different spin. It is worth mentioning that these equations can imply zero values of some amplitudes, reflecting the conservation laws of the model.

The last step of our procedure follows from the observation that due to R-parity [5], the "true" QCD processes with external gluons and quarks do not involve, in the tree approximation, any new interactions beyond the familiar quark and gluon couplings. Hence Eq.(11) allows us to express the QCD tree amplitudes in terms of the amplitudes for the processes involving the "exotic" particles ( preferably of spin zero ). Beyond the tree approximation things are more subtle. First of all one has to use a regularization procedure which preserves supersymmetry, like e.g. dimensional reduction [6]. Furthermore, the loops of "exotic" particles in diagrams with external gluons and quarks give some extra, undesired contributions, which have to be subtracted when calculating a "true" QCD amplitude.

In the rest of this paper we illustrate our procedure on a number of examples: the production, by gluon-gluon fusion, of two gluons, three gluons, and two gluinos plus a gluon, in tree approximation. Let $M( z^1_{s_1},..,z^k_{s_k};z^{k+1}_{s_{k+1}},..,z^n_{s_{n}} )$ denote the amplitude for the process with the initial particles $z^1,..,z^k$ of helicities $s_1,..,s_k$ and momenta $p_1,..,p_k$, and final particles $z^{k+1},..,z^n$ of helicities $s_{k+1},..,s_n$ and momenta $p_{k+1},..,p_n$, so that:

$$M( z^1_{s_1},...,z^k_{s_k};z^{k+1}_{s_{k+1}},...,z^n_{s_n} ) = \langle vac| z^n_{s_n} \dots z^{k+1}_{s_{k+1}} z^{k\dagger}_{s_k} \dots z^{1\dagger}_{s_1} |vac\rangle. \qquad (12)$$

For two gluons production, all helicity amplitudes can be obtained by crossing from $M( g^1_+,g^2_+ ; g^3_+,g^4_+ )$, $M( g^1_+,g^2_+ ; g^3_+,g^4_- )$ and $M( g^1_+,g^2_+ ; g^3_-,g^4_- )$. By inspecting the equations corresponding to:

$$\langle vac|[Q^1,\ g_-^4\ g_+^3\ g_+^{2\dagger}\ \lambda_+^{1\dagger}]|vac\rangle = 0, \tag{13}$$

$$\langle vac|[Q^1,\ g_-^4\ g_-^3\ g_+^{2\dagger}\ \lambda_+^{1\dagger}]|vac\rangle = 0, \tag{14}$$

we find

$$M(\ g_+^1,g_+^2\ ;\ g_+^3,g_-^4\ ) = M(\ g_+^1,g_+^2\ ;\ g_-^3,g_-^4\ ) = 0. \tag{15}$$

From

$$\langle vac|[Q^1,\ g_+^4\ g_+^3\ g_+^{2\dagger}\ \lambda_+^{1\dagger}]|vac\rangle = 0, \tag{16}$$

$$\langle vac|[Q^1,\ \lambda_+^4\ \lambda_+^3\ g_+^{2\dagger}\ \lambda_+^{1\dagger}]|vac\rangle = 0, \tag{17}$$

we obtain

$$\begin{aligned}&\Gamma_1^+ M(\ g_+^1,g_+^2\ ;\ g_+^3,g_+^4\ ) - \Gamma_3^+ M(\ \lambda_+^1,g_+^2\ ;\ \lambda_+^3,g_+^4\ )\\ &- \Gamma_4^+ M(\ \lambda_+^1,g_+^2\ ;\ g_+^3,\lambda_+^4\ ) = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}&-\Gamma_2^- M(\ \lambda_+^1,\lambda_+^2\ ;\ \lambda_+^3,\lambda_+^4\ ) + \Gamma_3^- M(\ \lambda_+^1,g_+^2\ ;\ g_+^3,\lambda_+^4\ )\\ &- \Gamma_4^- M(\ \lambda_+^1,g_+^2\ ;\ \lambda_+^3,g_+^4\ ) = 0.\end{aligned} \tag{19}$$

Straightforward calculation leads to the result that

$$|M(\ g_+^1,g_+^2\ ;\ g_+^3,g_+^4\ )| = |M(\ \lambda_+^1,\lambda_+^2\ ;\ \lambda_+^3,\lambda_+^4\ )|. \tag{20}$$

The phases of amplitudes are irrelevant, since the different helicity amplitudes do not interefere. These phases can be quite complicated because of the spin-statistics connection.

Similar manipulations with the second supersymmetry operator $Q^2$ allows us to express the r.h.s. of Eq.(20) in terms of the amplitude for the elastic scattering of gauge scalars. The result is

$$|M( g_+^1, g_+^2 \; ; \; g_+^3, g_+^4 )| = |M( \phi_+^1, \phi_+^2 \; ; \; \phi_+^3, \phi_+^4 )| . \tag{21}$$

It remains now to calculate $M( \phi_+^1, \phi_+^2 \; ; \; \phi_+^3, \phi_+^4 )$. It is easy to verify that, due to R-parity, the tree level diagrams with external gauge scalars ( and gluons ) involve only those interactions in the Lagrangian, Eq.(1), which correspond to a variant of the $[\Phi]^4$ theory:

$$\mathscr{L}_\phi = -\tfrac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - D_\mu \phi^\dagger \cdot D^\mu \phi - \tfrac{1}{2} g^2 (\phi \times \phi^\dagger) \cdot (\phi^\dagger \times \phi) , \tag{22}$$

where

$$D_\mu \phi = \partial_\mu \phi + g A_\mu \times \phi . \tag{23}$$

$F_{\mu\nu}$ denotes the gauge-field $A_\mu$ strength tensor. The cross product is taken with the gauge group structure constants $f_{XYZ}$.

There are three Feynman diagrams, see Fig. 1, which contribute to the amplitude:

$$M( \phi_+^1, \phi_+^2 \; ; \; \phi_+^3, \phi_+^4 ) = 2ig^2 f_{X13} f_{X24} \frac{(12)}{(13)} + [ \; 1 \leftrightarrow 2 \; ] . \tag{24}$$

We symbolically denoted the color index of the $i^{th}$ particle by i and $(ij) = p_i \cdot p_j$. After taking the square of the modulus of the amplitude, summing over final and averaging over initial color indices, we obtain:

$$|M(\ g_+^1,g_+^2\ ;\ g_+^3,g_+^4\ )|^2 = \frac{4g^4N^2}{N^2-1}\ s^2\ \{\ t^{-2} + u^{-2} + t^{-1}u^{-1}\ \}, \tag{25}$$

where s, t and u are the usual Mandelstam variables and N is the number of colors. All nonvanishing helicity amplitudes can be obtained from this equation by crossing. The final result is calculated by summing over final and averaging over initial polarizations:

$$|M(\ g^1,g^2\ ;\ g^3,g^4\ )|^2 = \frac{4g^4N^2}{N^2-1}\ \{\ 3 - \frac{st}{u^2} - \frac{tu}{s^2} - \frac{us}{t^2}\ \}, \tag{26}$$

in agreement with previous calculations.

For two gluon scattering into three gluons, we find, by essentially repeating the steps described in the previous example, the following results. All nonvanishing helicity amplitudes can be obtained by crossing from $M(\ g_+^1,g_+^2\ ;\ g_+^3,g_+^4,g_+^5\ )$ and $M(\ g_+^1,g_+^2\ ;\ g_-^3,g_+^4,g_+^5\ )$. Supersymmetry allows us to express these amplitudes in terms of the amplitudes for two gauge scalars scattering into two gauge scalars plus a gluon:

$$|M(\ g_+^1,g_+^2\ ;\ g_+^3,g_+^4,g_+^5\ )| = \frac{(12)}{(45)}\ |M(\ \phi_+^1,\phi_+^2\ ;\ g_+^3,\phi_+^4,\phi_+^5\ )|, \tag{27}$$

$$|M(\ g_+^1,g_+^2\ ;\ g_-^3,g_+^4,g_+^5\ )| = \frac{(45)}{(12)}\ |M(\ \phi_+^1,\phi_+^2\ ;\ g_-^3,\phi_+^4,\phi_+^5\ )|. \tag{28}$$

The calculation of the scalar amplitudes is simple and straightforward. Inserting this result ( summed and averaged over final and initial color indices, respectively ) into Eqs(27)-(28) we obtain

$$|M(\ g_+^1,g_+^2\ ;\ g_+^3,g_+^4,g_+^5\ )|^2 = \frac{-g^6N^3}{40(N^2-1)}\ (12)^4\ \left(\prod_{i<j}(ij)\right)^{-1} \cdot\sum_P(12)(23)(34)(45)(51), \tag{29}$$

and

$$|M(\ g_+^1,g_+^2\ ;\ g_-^3,g_+^4,g_+^5\ )|^2 = \frac{-g^6N^3}{40(N^2-1)}\ (45)^4\ \left(\prod_{i<j}(ij)\right)^{-1} \cdot\sum_P(12)(23)(34)(45)(51), \tag{30}$$

where $\sum_P$ denotes the sum over all permutations of 1 through 5. After summing over final and averaging over initial polarizations we get the final result:

$$|M(\ g^1,g^2\ ;\ g^3,g^4,g^5\ )|^2 = \frac{-g^6N^3}{240(N^2-1)}\ \sum_P(12)^4\ \left(\prod_{i<j}(ij)\right)^{-1} \cdot\sum_P(12)(23)(34)(45)(51), \tag{31}$$

in agreement with Ref.[7].

We used the algebraic manipulation program SCHOONSCHIP for book-keeping in the calculation of the scalar amplitudes. The number of terms generated at intermediate stages of the calculation never exceeded two hundred, compared to a typical brute force calculation which would produce around one million terms. Furthermore, our calculation "explains" the beautiful, factorized form of the result. The factor $\left(\prod(ij)\right)^{-1}\cdot\sum(12)(23)(34)(45)(51)$ is common to all helicity amplitudes, whereas $\sum(12)^4$ comes from summing over polarizations.

To further demonstrate the power of this method, we calculated the scattering of two gluons into two massless gluinos plus a gluon. The helicity amplitudes are related by equations similar to Eqs(27)-(28) and the matrix element squared suitable summed and averaged is

$$|M( g^1,g^2 \; ; \; g^3,\lambda^4,\lambda^5 )|^2 = \frac{-g^6N^3}{20(N^2-1)} \left( \prod_{i<j} (ij) \right)^{-1}$$

$$\cdot\sum_P (12)(23)(34)(45)(51)$$

$$\cdot\left\{(14)^3(15) + (24)^3(25) + (34)^3(35) + [\; 4\leftrightarrow 5 \;]\right\}. \qquad (32)$$

This is a new result.

We hope that these examples demonstrate the power, efficiency and elegance of the technique presented in this paper. A calculation of the scattering of two gluons into four gluons is currently under way using this technique.

We thank Keith Ellis for many useful discussions during the course of this work.

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FIGURE

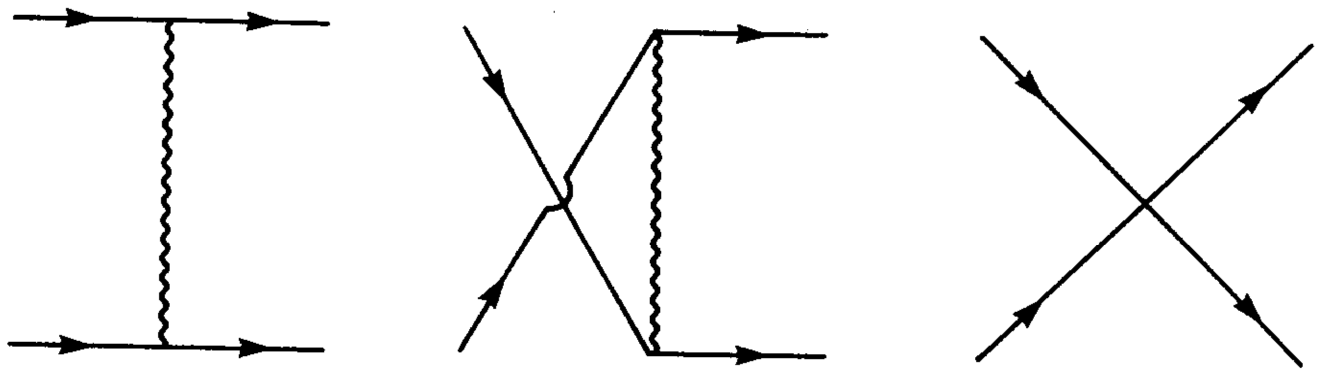

Fig.1.Feynman diagrams for elastic scattering of two scalars $\phi$.