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Cross section for hard processes involving two quarks and four gluons

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510
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The cross section for hard processes involving two quarks and four gluons is given in a form suitable for fast numerical calculations.

Four-jet spectroscopy at present (CERN SPS collider and Fermilab Tevatron) and future [Superconducting Super Collider (SSC)] hadron colliders holds great promise for testing QCD as well as for the discovery of new physics. Standard QCD interactions will cause significant backgrounds to many processes of interest, e.g., pair production of electroweak bosons.[1] Hence it is very important that we have a detailed understanding of conventional QCD four-jet production. In particular, knowledge of cross sections for hard-parton subprocesses is crucial for any reliable phenomenology of jet physics. In Ref. 2 we took one step in this direction, by presenting the cross section for four-gluon production by gluon-gluon fusion.

In this paper we give the cross section for hard processes involving two quarks and four gluons, in the tree approximation of QCD. The final cross section, which is presented in a form suitable for fast numerical evaluation, is applicable for the following processes: quark-antiquark annihilation into four gluons, quark-gluon inelastic scattering into a quark and three gluons, and gluon-gluon fusion into a quark-antiquark pair and two gluons. Guided by analyses of three-jet production (see, e.g., Ref. 1), we expect that for moderate values of the transverse momentum these processes are at least as important as the purely gluonic process. Since the masses of light quarks can be neglected at high-energy hadron colliders, we consider here the case of massless quarks.

To perform this calculation, we have further developed the techniques of Ref. 3, by explicitly using extended supersymmetry for a process involving quarks. This is the first time extended supersymmetry has been used for processes involving quarks as well as gluons. We adopt the convention that all particles involved in a scattering process are incoming. An outgoing particle of momentum p and helicity s will be represented as an incoming antiparticle of momentum $-p$ and helicity $-s$. Our convention is that the left-handed quark is represented by a spin-$\frac{1}{2}$ Weyl particle q, whereas the right-handed quark by a spin-$\frac{1}{2}$ Weyl particle $\bar{r}$. The left- and right-handed antiquarks are represented by r and $\bar{q}$, respectively. Let $M(z^1_{s_1},\ldots,z^n_{s_n})$ denote the amplitude for the process with the incoming particles $z^1,\ldots,z^n$ of helicities $s_1,\ldots,s_n$ and momenta $p_1,\ldots,p_n$. The momenta satisfy the conservation equation $\sum_{i=1}^{i=6}p_i=0$. We find that all nonvanishing quark-antiquark four-gluon helicity amplitudes can be obtained by crossing and/or complex conjugation from two amplitudes,

$$M(g^1_+,q^2_+,g^3_+,g^4_+,\bar{q}^5_-,g^6_-)$$

and

$$M(q^1_+,g^2_+,g^3_+,\bar{q}^4_-,g^5_-,g^6_-)\ .$$

These amplitudes can be expressed in terms of the amplitudes for processes involving spin-zero massless scalar gluons ϕ, spin-zero massless scalar quarks σ, left-handed quarks q, left-handed antiquarks r, and a smaller number of gluons, using supersymmetry relations (on-shell Ward-Takahashi identities). The first of two relations is very simple:

$$|M(g^1_+,q^2_+,g^3_+,g^4_+,\bar{q}^5_-,g^6_-)| = \frac{s_{56}}{\sqrt{|s_{23}s_{35}|}}\,|M(g^1_+,\sigma^2_+,\phi^3_+,g^4_+,\sigma^5_-,\phi^6_-)|\ , \quad (1)$$

where the Lorentz invariants s_{ij} are defined as usual, $s_{ij}=(p_i+p_j)^2$, and the scalar product is given by

$$p_ip_j=p_i^xp_j^x+p_i^yp_j^y+p_i^zp_j^z-E_iE_j \quad (2)$$

($E_i=p_i^t$; all particles are on mass shell, $p_ip_i=0$). The second supersymmetry relation is more complicated. However, it simplifes considerably in the c.m. system of particles 1 and 4:

$$\begin{aligned}|M(q^1_+,g^2_+,g^3_+,\bar{q}^4_-,g^5_-,g^6_-)| = \frac{1}{s_{23}s_{56}}\,\big|&(s_{12}+s_{13}+s_{23})^2M(q^1_+,\phi^2_+,\phi^3_+,\bar{q}^4_-,\phi^5_-,\phi^6_-)\\ &-2i\sqrt{-s_{14}}(s_{12}+s_{13}+s_{23})(p_5^x+p_6^x+ip_5^y+ip_6^y)M(\sigma^1_+,\phi^2_+,\phi^3_+,\sigma^4_-,\phi^5_-,\phi^6_-)\\ &-s_{14}(p_5^x+p_6^x+ip_5^y+ip_6^y)^2M(\bar{r}^1_+,\phi^2_+,\phi^3_+,r^4_-,\phi^5_-,\phi^6_-)\big|\ . \end{aligned} \quad (3)$$

Particles 1 and 4 are chosen to move along the negative and positive z axis, respectively.

We calculate the full cross section by first computing the amplitudes which occur on the right-hand side (RHS) of Eqs. (1) and (3), and subsequently using those equations in order to obtain the desired quark-antiquark four-gluon amplitudes. These amplitudes are calculated in the c.m. system of particles 1 and 4, and then reexpressed in terms of Lorentz invariants. Before presenting the result, let us adopt some convenient notation.

The squares of the absolute values of all helicity amplitudes will be generated from two generic functions,

$$B_0(p_1,p_2,p_3,p_4,p_5,p_6)$$

and

$$B_2(p_1,p_2,p_3,p_4,p_5,p_6)\ ,$$

defined as

$$B_0(p_1,p_2,p_3,p_4,p_5,p_6)=|M(q^1_+,g^2_+,g^3_+,\bar q^4_-,g^5_-,g^6_-)|^2\ ,$$
$$B_2(p_1,p_2,p_3,p_4,p_5,p_6)=|M(g^1_+,q^2_+,g^3_+,g^4_+,\bar q^5_-,g^6_-)|^2\ , \tag{4}$$

where the RHS implicitly contains the sum over the color indices of quarks and gluons, and the flavor indices of quarks. The square of the modulus of the invariant matrix element for quark-antiquark four-gluon process, averaged over initial colors, flavors, and polarizations, and summed over final colors, flavors, and polarizations, is given by

$$\begin{aligned}|M(q^1,\bar q^2,g^3,g^4,g^5,g^6)|^2=2^{1-G-Q}(N^2-1)^{-G}N^{-Q}N_f{}^{1-Q}\\ \times[B_0(134256)+B_0(135246)+B_0(136245)+B_0(154236)+B_0(164235)+B_0(234156)\\ +B_2(314526)+B_2(314625)+B_2(316524)+B_2(624513)\\ +B_2(614523)+B_2(623514)+B_2(324615)+B_2(324516)]\ ,\end{aligned} \tag{5}$$

where

$$B_{\binom{0}{2}}(ijklmn)=B_{\binom{0}{2}}(p_i,p_j,p_k,p_l,p_m,p_n)\ .$$

Here, N is the number of colors ($N=3$ for QCD), N_f is the number of light flavors, Q is the number of initial quarks and antiquarks, and G is the number of initial gluons. For example, the cross section for the annihilation of a quark with momentum p_1 and an antiquark with momentum p_2 into four gluons with momenta p_3,p_4,p_5,p_6 is obtained from Eq. (5) by setting $Q=2$, $G=0$, and replacing the momenta p_3,p_4,p_5,p_6 by $-p_3,-p_4,-p_5,-p_6$.

As the result of the computation of 232 Feynman diagrams, we obtain

$$B_0(p_1,p_2,p_3,p_4,p_5,p_6)=(\mathscr{D}_0^\dagger,\mathscr{D}_{0\pi}^\dagger)\begin{bmatrix}K & K_\rho\\ K_\rho & K\end{bmatrix}\begin{bmatrix}\mathscr{D}_0\\ \mathscr{D}_{0\pi}\end{bmatrix}\ ,$$
$$B_2(p_1,p_2,p_3,p_4,p_5,p_6)=(\mathscr{D}_2^\dagger,\mathscr{D}_{2\tau}^\dagger)\begin{bmatrix}K & K_\rho\\ K_\rho & K\end{bmatrix}\begin{bmatrix}\mathscr{D}_2\\ \mathscr{D}_{2\tau}\end{bmatrix}\ , \tag{6}$$

where $\mathscr{D}$, $\mathscr{D}_\pi$, and $\mathscr{D}_\tau$ are 16-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5, p_6, and K and K_ρ are constant, symmetric 16×16 matrices. The vectors $\mathscr{D}_\pi$ and $\mathscr{D}_\tau$ are obtained from the vectors $\mathscr{D}$ by the permutations ($p_2\leftrightarrow p_3$) and ($p_1\leftrightarrow p_4$), respectively, of the momentum variables in $\mathscr{D}$. The individual components of the vectors $\mathscr{D}$ represent the sums of all contributions proportional to the appropriately chosen 16 basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain four independent structures:

$$K=\frac{g^8(N^2-1)}{16N^3}(N^6K^{(6)}+N^4K^{(4)}+N^2K^{(2)}+K^{(0)})\ . \tag{7}$$

Here, g denotes the gauge coupling constant. The matrices $K^{(6)}$, $K^{(4)}$, $K^{(2)}$ and $K^{(0)}$ are given in Table I. The vectors $\mathscr{D}$ are related to the 22 diagrams D^F ($I=1$–22) involving left-handed quarks and four scalar gluons, 24 diagrams D^A ($I=1$–24) involving left-handed antiquarks and four scalar gluons, 22 diagrams D^S ($I=1$–22) involving scalar quarks and four scalar gluons, and 48 diagrams D^G ($I=1$–48) involving scalar quarks, two gluons, and two scalar gluons, in the following way:

$$\begin{aligned}\mathscr{D}_0&=\frac{1}{\sqrt{|s_{15}s_{45}|s_{23}s_{56}}}\\ &\quad\times[t_{123}^2C^FD^F+2t_{123}H(p_5+p_6,p_5)C^sD^s\\ &\quad+H(p_5+p_6,p_5)C^AD^A]\ ,\\ \mathscr{D}_2&=\frac{s_{56}}{2\sqrt{|s_{23}s_{35}|s_{14}}}C^GD^G\ ,\end{aligned} \tag{8}$$

where the constant matrices C^F (16×22), C^A (16×24), C^S (16×22), and C^G (16×48) are given in Table II. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij}=(p_i+p_j)^2$, $t_{ijk}=(p_i+p_j+p_k)^2$, and the complex function H is given by

$$\begin{aligned}H(p_i,p_j)=2[(p_1p_4)(p_ip_j)-(p_1p_i)(p_jp_4)\\ -(p_1p_j)(p_ip_4)+i\epsilon_{\mu\nu\rho\lambda}p_1^\mu p_i^\nu p_j^\rho p_4^\lambda]\ ,\end{aligned} \tag{9}$$

TABLE I. Matrices $K(I,J)$ ($I=1-16$, $J=1-16$).

Matrix $K^{(0)}$

−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Matrix $K^{(2)}$

3	2	−1	−2	1	0	−1	−1	1	1	1	1	0	0	0	0
2	3	−2	−1	0	1	−1	−1	1	1	1	1	0	0	0	0
−1	−2	3	2	1	0	1	1	−1	−1	−1	−1	0	0	0	0
−2	−1	2	3	0	1	1	1	−1	−1	−1	−1	0	0	0	0
1	0	1	0	3	2	1	1	−1	−1	−1	−1	0	0	0	0
0	1	0	1	2	3	1	1	−1	−1	−1	−1	0	0	0	0
−1	−1	1	1	1	1	2	2	0	0	0	0	0	0	0	0
−1	−1	1	1	1	1	2	2	0	0	0	0	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Matrix $K^{(4)}$

−3	−1	1	1	0	1	0	−1	−1	−1	−1	−1	0	0	1	−1
−1	−3	1	1	1	0	1	1	−1	−2	1	0	0	2	−1	−1
1	1	−3	−1	0	1	0	0	0	1	1	0	1	−1	0	0
1	1	−1	−3	1	0	0	−1	−1	1	0	0	−1	−1	0	2
0	1	0	1	−3	−1	−1	−1	2	1	0	−1	1	1	1	1
1	0	1	0	−1	−3	1	0	1	1	1	1	−1	1	−1	1
0	1	0	0	−1	1	−4	−2	1	2	−1	0	−1	−1	0	2
−1	1	0	−1	−1	0	−2	−4	2	1	0	−1	−1	−1	0	2
−1	−1	0	−1	2	1	1	2	−4	−2	0	2	0	0	−2	−2
−1	−2	1	1	1	1	2	1	−2	−4	2	0	2	2	−2	−2
−1	1	1	0	0	1	−1	0	0	2	−4	−2	−2	−2	2	2
−1	0	0	0	−1	1	0	−1	2	0	−2	−4	0	0	2	2
0	0	1	−1	1	−1	−1	−1	0	2	−2	0	−4	−4	2	2
0	2	−1	−1	1	1	−1	−1	0	2	−2	0	−4	−4	2	2
1	−1	0	0	1	−1	0	0	−2	−2	2	2	2	2	−4	−4
−1	−1	0	2	1	1	2	2	−2	−2	2	2	2	2	−4	−4

TABLE I. (*Continued*).

Matrix $K^{(6)}$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	1	0	0	0	−1	0
0	0	0	0	0	0	0	2	0	0	0	1	0	1	0	0
0	0	0	0	−1	0	0	0	2	0	0	0	−1	0	0	0
0	1	0	0	0	0	0	0	0	2	0	0	0	0	0	2
0	0	0	0	0	0	1	0	0	0	2	0	0	0	−2	0
0	0	0	0	0	0	0	1	0	0	0	2	0	1	0	0
0	0	0	0	0	0	0	0	−1	0	0	0	4	0	1	0
0	0	0	1	0	0	0	1	0	0	0	1	0	4	0	1
0	0	0	0	0	0	−1	0	0	0	−2	0	1	0	4	0
0	1	0	0	0	0	0	0	0	2	0	0	0	1	0	4

Matrix $K_\rho^{(0)}$

−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Matrix $K_\rho^{(2)}$

2	1	−2	−3	0	−1	−1	−1	1	1	1	1	0	0	0	0
1	2	−3	−2	−1	0	−1	−1	1	1	1	1	0	0	0	0
−2	−3	2	1	0	−1	1	1	−1	−1	−1	−1	0	0	0	0
−3	−2	1	2	−1	0	1	1	−1	−1	−1	−1	0	0	0	0
0	−1	0	−1	−2	−3	−1	−1	1	1	1	1	0	0	0	0
−1	0	−1	0	−3	−2	−1	−1	1	1	1	1	0	0	0	0
−1	−1	1	1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	1	1	−1	−1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE I. (*Continued*).

Matrix $K_\rho^{(4)}$

−1	0	1	0	0	0	0	1	0	0	−1	1	−1	−1	2	0
0	−1	0	1	0	0	−1	1	0	−1	−1	0	−1	1	0	0
1	0	−1	0	0	0	−1	−1	0	−1	2	1	2	0	−1	−1
0	1	0	−1	0	0	−1	−2	1	1	1	1	0	0	−1	1
0	0	0	0	1	0	2	1	−1	−1	−1	−1	1	−1	1	−1
0	0	0	0	0	1	1	1	1	0	−1	−2	1	1	1	1
0	−1	−1	−1	2	1	2	0	−1	−1	1	1	−2	−2	0	0
1	1	−1	−2	1	1	0	0	−1	1	−1	1	−2	−2	2	2
0	0	0	1	−1	1	−1	−1	2	0	1	1	0	2	−1	−1
0	−1	−1	1	−1	0	−1	1	0	0	−1	1	0	2	−1	−1
−1	−1	2	1	−1	−1	1	−1	1	−1	0	0	−2	0	1	1
1	0	1	1	−1	−2	1	1	1	1	0	2	−2	0	1	1
−1	−1	2	0	1	1	−2	−2	0	0	−2	−2	−2	−2	2	2
−1	1	0	0	−1	1	−2	−2	2	2	0	0	−2	−2	2	2
2	0	−1	−1	1	1	0	2	−1	−1	1	1	2	2	−2	−2
0	0	−1	1	−1	1	0	2	−1	−1	1	1	2	2	−2	−2

Matrix $K_\rho^{(6)}$

0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	0	0	0	1	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	−1	0	0	0	1	0	0	0	0	0	−1	0
0	0	0	1	0	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	1	0	0	0	0	0	−1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	−1	0	0	0	0	0	0	0	0	0	0	0	−1	0
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0	−1	0	0	0	2	0	0	0
0	0	0	0	0	0	0	2	0	0	0	1	0	2	0	0
0	0	1	0	0	0	−1	0	0	0	−1	0	0	0	2	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	2

where ϵ is the totally antisymmetric tensor, $\epsilon_{xyzt}=1$. For future use, we define one more function:

$$F(p_i,p_j)=[(p_1p_4)(p_ip_j)+(p_1p_i)(p_jp_4)-(p_1p_j)(p_ip_4)]/(p_1p_4)\ . \tag{10}$$

Note that when evaluating B_0 and B_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions H and F on the momenta p_1 and p_4.

The diagrams D^F are

$$D^F(1)=\frac{4}{s_{12}s_{46}t_{125}}\{F(p_5,p_6)H(p_2,p_5)-F(p_2,p_6)H(p_5,p_5)+[F(p_2,p_5)+s_{12}]H(p_6,p_5)\}\ ,$$

$$D^F(2)=\frac{4}{s_{12}s_{45}t_{126}}\{F(p_6,p_5)H(p_2,p_5)-F(p_2,p_5)H(p_6,p_5)+[F(p_2,p_6)+s_{12}]H(p_5,p_5)\}\ ,$$

$$D^F(3)=\frac{-4}{s_{12}s_{36}s_{45}}\left[\left[F(p_2,p_6)+\frac{s_{12}}{2}\right]H(p_5,p_5)+\left[F(p_6,p_5)+\frac{s_{45}}{2}\right]H(p_2,p_5)-F(p_2,p_5)H(p_6,p_5)\right]\ ,$$

$$D^F(4)=\frac{-4}{s_{12}s_{35}s_{46}}\left[\left[F(p_2,p_5)+\frac{s_{12}}{2}\right]H(p_6,p_5)+\left[F(p_5,p_6)+\frac{s_{46}}{2}\right]H(p_2,p_5)-F(p_2,p_6)H(p_5,p_5)\right]\ ,$$

TABLE II. Matrices $C^F(I,J)$ ($I=1-16$, $J=1-22$), $C^A(I,J)$ ($I=1-16$, $J=1-24$), $C^S(I,J)$ ($I=1-16$, $J=1-22$), and $C^G(I,J)$ ($I=1-16$, $J=1-48$). Indices I and J specify row numbers and column numbers, respectively.

Matrix C^F

0	0	0	1	0	1	0	0	0	0	0	0	−1	0	0	0	0	−1	0	0	0	0
0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	−1	0	−1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	−1	0	0	0	−1	0	0	0	0	−1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	0	0	−1	0	−1	0	1	0	0	0	1	1	1	0	1	1	1	0	0	−1	1
0	−1	1	0	1	0	0	0	−1	0	0	−1	−1	0	1	−1	−1	−1	0	0	1	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	−1	−1	0	0	0	1	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	1	0	0	0	−1	1
0	0	0	0	0	0	−1	0	0	0	−1	0	0	0	0	0	0	0	0	−1	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−2
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	−1	0	0	1	0	−1	−1
0	0	0	0	−1	0	0	0	0	−1	0	0	0	0	−1	1	0	0	−1	0	1	1

Matrix C^A

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0
0	0	1	0	0	0	0	−1	0	0	0	0	1	0	0	0	0	0	0	−1	0	0	0	0
0	0	0	0	0	−1	0	0	0	−1	0	0	−1	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	0	0	−1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1	−1	−1	0	0	1	−1
0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	−1	0	−1	−1	0	0	0	1	−1
−1	0	0	0	0	0	1	0	0	0	0	−1	0	0	0	0	−1	1	1	0	0	0	−1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	2
0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	1	0	−2
0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	0	−1	0	0	1	0	−1	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	0	0	−1	0	1	1

TABLE II. (*Continued*).

Matrix C^S

0	0	0	0	0	1	0	1	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	−1	0	−1	1	1	0	0	0	0	0	0	−2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	−1	0	−1	1	1	0	0	−4	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	−1	−1	0	0	0	0	0	0	2	−4	0	0	0	0
0	0	0	0	0	1	0	−1	−1	1	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	0	−1	0	1	1	−1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	−2	1	−2	1	1	−1	−2	0	0	0	0	0	0	1	−1	1	−1
0	0	0	0	−1	2	−1	2	−1	−1	0	0	0	0	−1	0	2	0	1	1	−1	1
0	0	0	0	−1	1	1	−1	0	0	1	0	0	0	0	0	0	0	−1	1	−1	1
0	0	0	0	1	−1	−1	1	0	0	0	0	0	0	1	−2	0	0	−1	−1	1	−1
0	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−2	0
1	−1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	−1	−1	−1	1
−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1	1	1	−1

Matrix C^G (columns $J=1–24$)

1	0	0	0	1	0	1	−1	−1	−1	−1	1	1	1	0	0	1	0	0	0	−1	0	1	0
−1	0	0	0	−1	0	−1	1	1	1	1	−1	−1	−1	0	0	−1	0	0	0	1	0	−1	0
0	0	1	0	1	0	−1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1	0	0
0	0	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	1	0	0
0	0	0	0	0	0	0	1	1	1	1	−1	−1	−1	0	0	0	0	0	0	−1	−1	−1	0
0	0	0	0	0	0	0	−1	−1	−1	−1	1	1	1	0	0	0	0	0	0	1	1	1	0
0	0	0	0	0	0	0	−1	−1	1	−1	−1	0	1	−1	0	0	0	0	0	0	0	−1	0
0	0	0	0	0	0	0	1	1	−1	1	1	0	−1	1	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	2	2	0	0	−1	0	−1	0	0	0	0	0	0	−1	−1
0	0	0	0	0	0	0	0	0	−2	−2	0	0	1	0	1	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	1	1	1	−1	1	0	0	1	1	0	0	0	0	0	0	0	−1
0	0	0	0	0	0	0	−1	−1	−1	1	−1	0	0	−1	−1	0	0	0	0	0	0	0	1
0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0

TABLE II. (*Continued*).

Matrix C^G (columns $J=25-48$)

1	0	0	0	0	0	−1	0	0	0	1	0	0	0	1	0	0	0	0	0	−1	0	0	0
−1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	−1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
1	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	−1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	1	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
−2	0	1	1	0	1	1	0	0	−1	0	0	0	0	0	0	0	0	0	1	1	0	0	−1
2	0	−1	−1	0	−1	−1	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	−1
0	−1	−1	−1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	1	1	1	0	0	0	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	−1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
0	0	−1	−2	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1
0	0	1	2	0	0	0	0	0	−1	0	0	0	1	0	0	0	1	0	0	0	0	0	1

$$D^F(5)=\frac{4}{s_{36}s_{45}t_{136}}\left[F(p_6,p_5)H(p_3,p_5)-F(p_3,p_5)H(p_6,p_5)-\left[F(p_6,p_3)-\frac{s_{36}}{2}-\frac{s_{13}}{2}+\frac{s_{16}}{2}\right]H(p_5,p_5)\right],$$

$$D^F(6)=\frac{4}{s_{35}s_{46}t_{135}}\left[F(p_5,p_6)H(p_3,p_5)-F(p_3,p_6)H(p_5,p_5)-\left[F(p_5,p_3)-\frac{s_{35}}{2}-\frac{s_{13}}{2}+\frac{s_{15}}{2}\right]H(p_6,p_5)\right],$$

$$D^F(7)=\frac{4}{s_{25}s_{46}t_{125}}\left[F(p_5,p_6)H(p_2,p_5)-F(p_2,p_6)H(p_5,p_5)-\left[F(p_5,p_2)-\frac{s_{25}}{2}-\frac{s_{12}}{2}+\frac{s_{15}}{2}\right]H(p_6,p_5)\right],$$

$$D^F(8)=\frac{4}{s_{12}s_{36}t_{125}}\left[F(p_2,p_6)H(p_3,p_5)-F(p_2,p_3)H(p_6,p_5)-\left[F(p_3,p_6)+\frac{s_{46}}{2}-\frac{s_{34}}{2}-\frac{s_{36}}{2}\right]H(p_2,p_5)\right],$$

$$D^F(9)=\frac{4}{s_{12}s_{35}t_{126}}\left[F(p_2,p_5)H(p_3,p_5)-F(p_2,p_3)H(p_5,p_5)-\left[F(p_3,p_5)+\frac{s_{45}}{2}-\frac{s_{34}}{2}-\frac{s_{35}}{2}\right]H(p_2,p_5)\right], \qquad (11)$$

$$D^F(10)=\frac{2}{s_{36}s_{45}t_{145}}(s_{26}+s_{36}-s_{23})H(p_5,p_5),\quad D^F(11)=\frac{2}{s_{25}s_{46}t_{146}}(s_{25}+s_{35}-s_{23})H(p_6,p_5),$$

$$D^F(12)=\frac{2}{s_{12}s_{36}t_{124}}(s_{56}-s_{35}-s_{36})H(p_2,p_5),\quad D^F(13)=\frac{2}{s_{12}s_{35}t_{124}}(s_{56}-s_{36}-s_{35})H(p_2,p_5),$$

$$D^F(14)=\frac{4}{s_{25}s_{36}t_{125}}\left[\left[F(p_5,p_6)+\frac{t_{125}}{4}\right]H(p_2,p_5)-\left[F(p_2,p_6)+\frac{t_{125}}{4}\right]H(p_5,p_5)\right.$$
$$\left.-\left[F(p_5,p_2)+\frac{s_{15}}{2}-\frac{s_{12}}{2}-\frac{s_{25}}{2}\right]H(p_6,p_5)\right],$$

$$D^F(15)=\frac{4}{s_{25}s_{36}t_{136}}\left[\left[F(p_6,p_5)+\frac{t_{136}}{4}\right]H(p_3,p_5)-\left[F(p_3,p_5)+\frac{t_{136}}{4}\right]H(p_6,p_5)\right.$$
$$\left.-\left[F(p_6,p_3)+\frac{s_{16}}{2}-\frac{s_{13}}{2}-\frac{s_{36}}{2}\right]H(p_5,p_5)\right],$$

$$D^F(16)=\frac{1}{s_{14}s_{25}s_{36}}[(s_{23}-s_{26}+s_{35}-s_{56})H(p_5-p_2,p_5)+(s_{23}-s_{35}+s_{26}-s_{56})H(p_3-p_6,p_5)$$
$$+(s_{23}-s_{26}-s_{35}+s_{56})H(p_2+p_5,p_5)],$$

$$D^F(17)=\frac{2}{s_{14}s_{36}t_{124}}(s_{56}-s_{35}-s_{36})H(p_2,p_5),\quad D^F(18)=\frac{2}{s_{14}s_{35}t_{124}}(s_{56}-s_{35}-s_{36})H(p_2,p_5),$$

$$D^F(19)=\frac{-2}{s_{14}s_{36}t_{145}}(s_{23}-s_{26}-s_{36})H(p_5,p_5),\quad D^F(20)=\frac{-2}{s_{14}s_{25}t_{146}}(s_{23}-s_{35}-s_{25})H(p_6,p_5),$$

$$D^F(21)=\frac{1}{s_{14}s_{36}}H(p_3-p_6,p_5),\quad D^F(22)=\frac{1}{s_{14}s_{25}}H(p_2-p_5,p_5).$$

The diagrams D^A are

$$D^A(1)=\frac{4}{s_{16}s_{24}t_{136}}\{F(p_3,p_2)H(p_5+p_6,p_6)-F(p_6,p_2)H(p_5+p_6,p_3)+[F(p_6,p_3)+s_{16}]H(p_5+p_6,p_2)\},$$

$$D^A(2)=\frac{4}{s_{16}s_{34}t_{126}}\{F(p_2,p_3)H(p_5+p_6,p_6)-F(p_6,p_3)H(p_5+p_6,p_2)+[F(p_6,p_2)+s_{16}]H(p_5+p_6,p_3)\},$$

$$D^A(3)=\frac{4}{s_{15}s_{24}t_{135}}\{F(p_3,p_2)H(p_5+p_6,p_5)-F(p_5,p_2)H(p_5+p_6,p_3)+[F(p_5,p_3)+s_{15}]H(p_5+p_6,p_2)\},$$

$$D^A(4)=\frac{4}{s_{15}s_{34}t_{125}}\{F(p_2,p_3)H(p_5+p_6,p_5)-F(p_5,p_3)H(p_5+p_6,p_2)+[F(p_5,p_2)+s_{15}]H(p_5+p_6,p_3)\},$$

$$D^A(5)=\frac{4}{s_{15}s_{24}s_{36}}\left[F(p_5,p_2)H(p_5+p_6,p_6)-\left[F(p_5,p_6)+\frac{s_{15}}{2}\right]H(p_5+p_6,p_2)-\left[F(p_6,p_2)+\frac{s_{24}}{2}\right]H(p_5+p_6,p_5)\right],$$

$$D^A(6)=\frac{4}{s_{16}s_{24}s_{35}}\left[F(p_6,p_2)H(p_5+p_6,p_5)-\left[F(p_6,p_5)+\frac{s_{16}}{2}\right]H(p_5+p_6,p_2)-\left[F(p_5,p_2)+\frac{s_{24}}{2}\right]H(p_5+p_6,p_6)\right],$$

$$D^A(7)=\frac{4}{s_{24}s_{36}t_{136}}\left[F(p_6,p_2)H(p_5+p_6,p_3)-F(p_3,p_2)H(p_5+p_6,p_6)-\left[F(p_6,p_3)-\frac{s_{36}}{2}-\frac{s_{13}}{2}+\frac{s_{16}}{2}\right]H(p_5+p_6,p_2)\right],$$

$$D^A(8)=\frac{4}{s_{24}s_{35}t_{135}}\left[F(p_5,p_2)H(p_5+p_6,p_3)-F(p_3,p_2)H(p_5+p_6,p_5)-\left[F(p_5,p_3)-\frac{s_{35}}{2}-\frac{s_{13}}{2}+\frac{s_{15}}{2}\right]H(p_5+p_6,p_2)\right],$$

$$D^A(9)=\frac{4}{s_{15}s_{36}t_{125}}\left[F(p_5,p_6)H(p_5+p_6,p_3)-F(p_5,p_3)H(p_5+p_6,p_6)-\left[F(p_3,p_6)+\frac{s_{46}}{2}-\frac{s_{34}}{2}-\frac{s_{36}}{2}\right]H(p_5+p_6,p_5)\right],$$

$$\begin{aligned}D^A(10)=\frac{4}{s_{16}s_{35}t_{126}}\Bigg[&F(p_6,p_5)H(p_5+p_6,p_3)-F(p_6,p_3)H(p_5+p_6,p_5)\\&-\left[F(p_3,p_5)+\frac{s_{45}}{2}-\frac{s_{34}}{2}-\frac{s_{35}}{2}\right]H(p_5+p_6,p_6)\Bigg],\end{aligned} \tag{12}$$

$$\begin{aligned}D^A(11)=\frac{4}{s_{16}s_{25}t_{136}}\Bigg[&F(p_6,p_5)H(p_5+p_6,p_2)-F(p_6,p_2)H(p_5+p_6,p_5)\\&-\left[F(p_2,p_5)+\frac{s_{45}}{2}-\frac{s_{24}}{2}-\frac{s_{25}}{2}\right]H(p_5+p_6,p_6)\Bigg],\end{aligned}$$

$$D^A(12)=\frac{2}{s_{24}s_{36}t_{124}}(s_{35}+s_{36}-s_{56})H(p_5+p_6,p_2),\quad D^A(13)=\frac{2}{s_{24}s_{35}t_{124}}(s_{35}+s_{36}-s_{56})H(p_5+p_6,p_2),$$

$$D^A(14)=\frac{2}{s_{15}s_{36}t_{145}}(s_{23}-s_{26}-s_{36})H(p_5+p_6,p_5),\quad D^A(15)=\frac{2}{s_{16}s_{25}t_{146}}(s_{23}-s_{35}-s_{25})H(p_5+p_6,p_6),$$

$$\begin{aligned}D^A(16)=\frac{4}{s_{25}s_{36}t_{125}}\Bigg[&\left[F(p_5,p_6)+\frac{t_{125}}{4}\right]H(p_5+p_6,p_2)-\left[F(p_2,p_6)+\frac{t_{125}}{4}\right]H(p_5+p_6,p_5)\\&-\left[F(p_5,p_2)+\frac{s_{15}}{2}-\frac{s_{12}}{2}-\frac{s_{25}}{2}\right]H(p_5+p_6,p_6)\Bigg],\end{aligned}$$

$$\begin{aligned}D^A(17)=\frac{4}{s_{25}s_{36}t_{136}}\Bigg[&\left[F(p_6,p_5)+\frac{t_{136}}{4}\right]H(p_5+p_6,p_3)-\left[F(p_3,p_5)+\frac{t_{136}}{4}\right]H(p_5+p_6,p_6)\\&-\left[F(p_6,p_3)+\frac{s_{16}}{2}-\frac{s_{13}}{2}-\frac{s_{36}}{2}\right]H(p_5+p_6,p_5)\Bigg],\end{aligned}$$

$$\begin{aligned}D^A(18)=\frac{1}{s_{25}s_{36}s_{14}}[&(s_{23}-s_{26}+s_{35}-s_{56})H(p_5+p_6,p_5-p_2)+(s_{23}-s_{35}+s_{26}-s_{56})H(p_5+p_6,p_3-p_6)\\&+(s_{23}-s_{26}-s_{35}+s_{56})H(p_5+p_6,p_2+p_5)],\end{aligned}$$

$$D^A(19)=\frac{2}{s_{14}s_{36}t_{124}}(s_{56}-s_{35}-s_{36})H(p_5+p_6,p_2),\quad D^A(20)=\frac{2}{s_{14}s_{35}t_{124}}(s_{56}-s_{35}-s_{36})H(p_5+p_6,p_2),$$

$$D^A(21)=\frac{2}{s_{14}s_{36}t_{145}}(s_{26}+s_{36}-s_{23})H(p_5+p_6,p_5),\quad D^A(22)=\frac{2}{s_{14}s_{25}t_{146}}(s_{25}+s_{35}-s_{23})H(p_5+p_6,p_6),$$

$$D^A(23)=\frac{1}{s_{14}s_{36}}H(p_5+p_6,p_3-p_6),\quad D^A(24)=\frac{1}{s_{14}s_{25}}H(p_5+p_6,p_2-p_5).$$

The diagrams D^S are

$$D^S(1)=\frac{1}{s_{14}s_{36}t_{145}}(s_{45}-s_{15})(s_{23}-s_{26}-s_{36}),\quad D^S(2)=\frac{1}{s_{36}t_{145}}(s_{23}-s_{26}-s_{36}),$$

$$D^S(3)=\frac{1}{s_{14}s_{25}t_{146}}(s_{46}-s_{16})(s_{23}-s_{35}-s_{25}),\quad D^S(4)=\frac{1}{s_{25}t_{146}}(s_{23}-s_{35}-s_{25}),$$

$$D^s(5)=\frac{1}{s_{14}s_{36}t_{124}}(s_{12}-s_{24})(s_{56}-s_{35}),\quad D^S(6)=\frac{1}{s_{14}t_{124}}(s_{12}-s_{24}),\quad D^S(7)=\frac{1}{s_{36}t_{124}}(s_{56}-s_{35})\ ,$$

$$D^S(8)=\frac{1}{t_{124}},\quad D^S(9)=\frac{1}{s_{14}s_{35}t_{124}}(s_{12}-s_{24})(s_{56}-s_{36}),\quad D^S(10)=\frac{1}{s_{35}t_{124}}(s_{56}-s_{36})\ ,$$

$$D^S(11)=\frac{1}{s_{25}s_{36}t_{125}}(s_{12}-s_{15}-s_{25})(s_{46}-s_{34}+s_{36}),\quad D^S(12)=\frac{1}{s_{36}t_{125}}(s_{46}-s_{34}+s_{36})\ , \tag{13}$$

$$D^S(13)=\frac{1}{t_{125}},\quad D^S(14)=\frac{1}{s_{35}t_{135}}(s_{13}-s_{15}-s_{35}),\quad D^S(15)=\frac{1}{s_{25}s_{36}t_{136}}(s_{13}-s_{16}-s_{36})(s_{45}-s_{24}+s_{25})\ ,$$

$$D^S(16)=\frac{1}{s_{36}t_{136}}(s_{13}-s_{16}-s_{36}),\quad D^S(17)=\frac{1}{s_{35}t_{126}}(s_{45}-s_{34}+s_{35}),\quad D^S(18)=\frac{1}{t_{126}}\ ,$$

$$D^S(19)=\frac{1}{2s_{25}s_{36}}(s_{23}-s_{26}-s_{35}+s_{56}),\quad D^S(20)=\frac{1}{2s_{14}s_{36}}(s_{13}-s_{16}-s_{34}+s_{46}),\quad D^S(21)=\frac{1}{2s_{14}s_{25}}(s_{12}-s_{24}-s_{15}+s_{45})\ ,$$

$$D^S(22)=\frac{1}{2s_{14}s_{25}s_{36}}[(s_{12}-s_{24}-s_{15}+s_{45})(s_{16}-s_{13}+s_{46}-s_{34})+(s_{23}-s_{26}-s_{35}+s_{56})(s_{24}-s_{12}+s_{45}-s_{15})$$
$$+(s_{13}-s_{16}-s_{34}+s_{46})(s_{35}-s_{23}+s_{56}-s_{26})]\ .$$

The diagrams D^G are

$$D^G(1)=\frac{2}{s_{36}t_{124}}(s_{35}-s_{56})(s_{12}-s_{24}+s_{14}),\quad D^G(2)=\frac{2}{s_{25}t_{134}}(s_{26}-s_{56})(s_{13}-s_{34}+s_{14})\ ,$$

$$D^G(3)=\frac{2}{s_{36}t_{145}}(s_{26}-s_{23})(s_{45}-s_{15}+s_{14}),\quad D^G(4)=\frac{2}{s_{25}t_{146}}(s_{35}-s_{23})(s_{46}-s_{16}+s_{14})\ ,$$

$$D^G(5)=\frac{2}{s_{36}}(p_1-p_4)(p_3-p_6),\quad D^G(6)=\frac{2}{s_{25}}(p_1-p_4)(p_2-p_5)\ ,$$

$$D^G(7)=\frac{4}{s_{25}s_{36}}\{[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)]-[(p_2-p_5)(p_3+p_6)][(p_1-p_4)(p_3-p_6)]$$
$$+[(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)]\}\ ,$$

$$D^G(8)=\frac{4}{s_{25}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]H(p_2,p_3)-[(p_1+p_2-p_5)(p_4-p_3+p_6)]H(p_2,p_6)$$
$$-[(p_1-p_2+p_5)(p_4+p_3-p_6)]H(p_5,p_3)+[(p_1-p_2+p_5)(p_4-p_3+p_6)]H(p_5,p_6)$$
$$-[p_1(p_2-p_5)]H(p_3-p_6,p_3+p_6)-[p_4(p_3-p_6)]H(p_2+p_5,p_2-p_5)$$
$$+2s_{14}[p_1(p_2-p_5)][p_4(p_3-p_6)]\}\ ,$$

$$D^G(9)=\frac{2}{s_{25}s_{36}}\{s_{14}[(p_2-p_5)(p_3-p_6)]-H(p_3-p_6,p_2-p_5)+H(p_2-p_5,p_3-p_6)\}\ ,$$

$$D^G(10)=\frac{2s_{14}}{t_{125}},\quad D^G(11)=\frac{2}{s_{25}t_{125}}[s_{14}(s_{12}-s_{15})-H(p_2+p_5,p_2-p_5)]\ ,$$

$$D^G(12)=\frac{2}{s_{36}t_{125}}[s_{14}(s_{46}-s_{34})+H(p_3-p_6,p_3+p_6)]\ ,$$

$$D^G(13)=\frac{4}{s_{12}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4-p_3+p_6)]H(p_2,p_6)-[(p_1+p_2-p_5)(p_4+p_3-p_6)]H(p_2,p_3)$$
$$+[p_4(p_3-p_6)]H(p_2,p_2-p_5)\}\ ,$$

$$D^G(14)=\frac{4}{s_{25}s_{34}t_{125}}\{[(p_1-p_2+p_5)(p_4+p_3-p_6)]H(p_5,p_3)-[(p_1+p_2-p_5)(p_4+p_3-p_6)]H(p_2,p_3)$$
$$+[p_1(p_2-p_5)]H(p_3-p_6,p_3)\}\ ,$$

$$D^G(15)=\frac{4}{s_{15}s_{36}t_{125}}\{[(p_1-p_2+p_5)(p_4-p_3+p_6)]H(p_5,p_6)-[(p_1-p_2+p_5)(p_4+p_3-p_6)]H(p_5,p_3)$$
$$-[p_4(p_3-p_6)]H(p_5,p_2-p_5)\}\ ,$$

$$D^G(16)=\frac{4}{s_{25}s_{46}t_{125}}\{[(p_1-p_2+p_5)(p_4-p_3+p_6)]H(p_5,p_6)-[(p_1+p_2-p_5)(p_4-p_3+p_6)]H(p_2,p_6)$$
$$-[p_1(p_2-p_5)]H(p_3-p_6,p_6)\}\ ,$$

$$D^G(17)=\frac{4}{s_{12}s_{36}t_{124}}(s_{56}-s_{35})H(p_2,p_2),\quad D^G(18)=\frac{4}{s_{45}s_{36}t_{145}}(s_{23}-s_{26})H(p_5,p_5)\ , \tag{14}$$

$$D^G(19)=\frac{4}{s_{13}s_{25}t_{134}}(s_{56}-s_{26})H(p_3,p_3),\quad D^G(20)=\frac{4}{s_{25}s_{46}t_{146}}(s_{23}-s_{35})H(p_6,p_6)\ ,$$

$$D^G(21)=\frac{2}{s_{12}s_{36}}H(p_2,p_3-p_6),\quad D^G(22)=\frac{-2}{s_{36}s_{45}}H(p_3-p_6,p_5),\quad D^G(23)=\frac{2}{s_{34}t_{125}}H(p_3-p_6,p_3)\ ,$$

$$D^G(24)=\frac{-2}{s_{46}t_{125}}H(p_3-p_6,p_6),\quad D^G(25)=\frac{2}{s_{12}t_{125}}H(p_2,p_2-p_5),\quad D^G(26)=\frac{-2}{s_{15}t_{125}}H(p_5,p_2-p_5)\ ,$$

$$D^G(27)=\frac{2}{s_{13}s_{25}}H(p_3,p_2-p_5),\quad D^G(28)=\frac{-2}{s_{25}s_{46}}H(p_2-p_5,p_6),\quad D^G(29)=\frac{8}{s_{12}s_{36}s_{45}}[(p_3-p_6)(p_4+p_5)]H(p_2,p_5)\ ,$$

$$D^G(30)=\frac{4}{s_{12}s_{46}t_{125}}[(p_1+p_2-p_5)(p_4-p_3+p_6)]H(p_2,p_6),\quad D^G(31)=\frac{4}{s_{12}s_{34}t_{125}}[(p_1+p_2-p_5)(p_4+p_3-p_6)]H(p_2,p_3)\ ,$$

$$D^G(32)=\frac{4}{s_{15}s_{34}t_{125}}[(p_1-p_2+p_5)(p_4+p_3-p_6)]H(p_5,p_3),\quad D^G(33)=\frac{4}{s_{15}s_{46}t_{125}}[(p_1-p_2+p_5)(p_4-p_3+p_6)]H(p_5,p_6)\ ,$$

$$D^G(34)=\frac{8}{s_{13}s_{25}s_{46}}[(p_2-p_5)(p_4+p_6)]H(p_3,p_6),\quad D^G(35)=\frac{2}{t_{124}}(s_{24}-s_{12}-s_{14})\ ,$$

$$D^G(36)=\frac{2}{t_{134}}(s_{34}-s_{13}-s_{14}),\quad D^G(37)=\frac{2}{t_{145}}(s_{15}-s_{45}-s_{14}),\quad D^G(38)=\frac{2}{t_{146}}(s_{16}-s_{46}-s_{14})\ ,$$

$$D^G(39)=\frac{4}{s_{12}t_{124}}H(p_2,p_2),\quad D^G(40)=\frac{4}{s_{45}t_{145}}H(p_5,p_5),\quad D^G(41)=\frac{4}{s_{13}t_{134}}H(p_3,p_3),\quad D^G(42)=\frac{4}{s_{46}t_{146}}H(p_6,p_6)\ ,$$

$$D^G(43)=\frac{-4}{s_{12}s_{45}}H(p_2,p_5),\quad D^G(44)=\frac{-4}{s_{12}s_{46}}H(p_2,p_6),\quad D^G(45)=\frac{4}{s_{12}s_{34}}H(p_2,p_3)\ ,$$

$$D^G(46)=\frac{-4}{s_{15}s_{34}}H(p_5,p_3),\quad D^G(47)=\frac{4}{s_{15}s_{46}}H(p_5,p_6),\quad D^G(48)=\frac{-4}{s_{13}s_{46}}H(p_3,p_6)\ .$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using Eqs. (11)–(14). The result is substituted into Eq. (8) to obtain the vectors $\mathscr{D}_0$ and $\mathscr{D}_2$. After generating the vectors $\mathscr{D}_{0\pi}$ and $\mathscr{D}_{2\tau}$ by the appropriate permutations of momenta, Eq. (6) is used to obtain the functions B_0 and B_2. Finally, the total cross section is calculated by using Eq. (5). The FORTRAN5 program based on such a scheme generates two Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

The following testing procedures can be used in the numerical calculations based on the algorithm presented in this paper. First, the function $B_2(p_1,p_2,p_3,p_4,p_5,p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1,p_3,p_4). Another, very important test examines the singular behavior of the cross section in the kinematical limit of any two partons i and j moving collinearly, i.e., with s_{ij} going to zero. The double poles of the form $(s_{ij})^{-2}$ should be absent and, further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the appropriate two goes to three cross section,[4] convoluted with the Altarelli-Parisi probability[5] for the decay of the final particles into partons i and j. For example, when the quark and antiquark momenta become parallel, the invariant quark-antiquark four-gluon matrix element squared must factorize into the five-gluon matrix element squared and the Altarelli-Parisi probability for gluon branching into a quark-antiquark pair. It is worth mentioning that, similar to the case of the lepton-antilepton four-parton matrix element squared,[6] the factorization holds only after averaging over the azimuthal angle of the branching process plane. Our result has successfully passed both these numerical checks.

While this manuscript was in preparation, we have learned from Z. Kunszt, that he has also completed a numerical routine for this cross section[7] using a different set of techniques. Together, we have made a numerical comparison of the two results and complete agreement was found.

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