Bootstrapping One-Loop Amplitudes

Or: Needles and Large HaystaCks

Carola F. Berger

Stanford Linear Accelerator Center

in collaboration with

Zvi Bern, Lance Dixon, Darren Forde, David Kosower
and Vittorio Del Duca

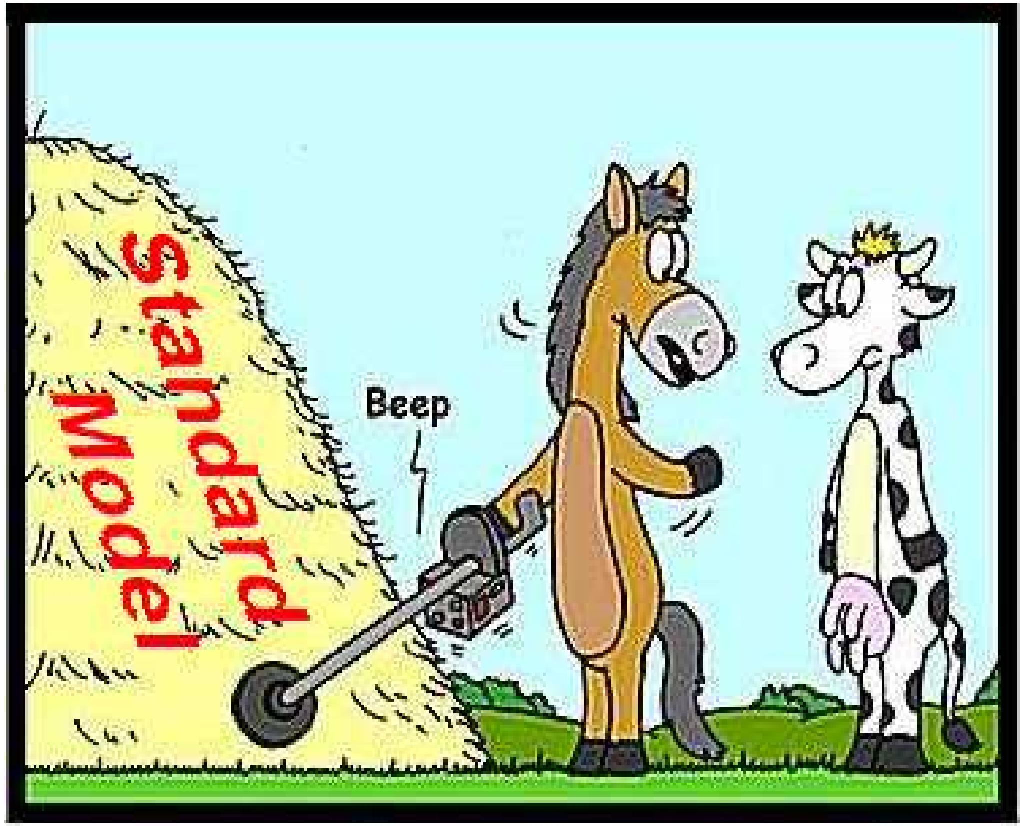
Fermilab – Dec. 7th, 2006

Outline

- **Introduction to precision calculations relevant for the LHC:**
  Why we (= theorists) haven't fulfilled the experimenters' wishlist yet
- **A new method: On-shell recursion relations**
  But: QCD at one loop is not so simple...
- **On-shell bootstrap at one loop**
- **Summary, open questions, and outlook**
  The wishlist will get done

Particle Physics in the 20th Century

What the LEP, Tevatron, . . . told us



You were right: There's a needle in this haystack...

NLO Calculations are Needed!

Physics at the LHC
- Particle Physics in the 20th Century
- NLO Calculations are Needed!
- Particle Physics in the 21st Century?
- The LHC Wishlists
- Precision Calculations
- Feynman Graphs

On-Shell Recursion Relations at Tree Level

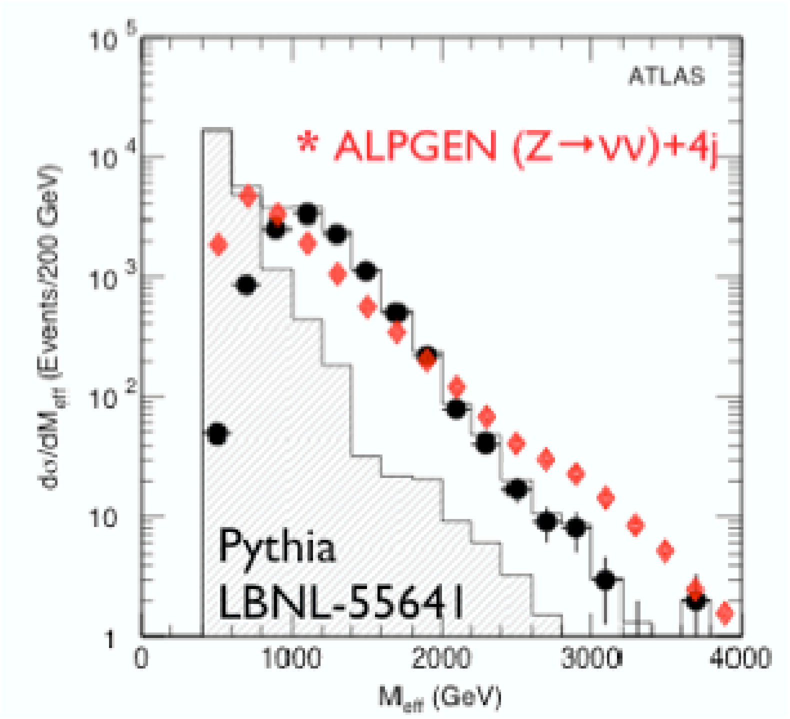
The Bootstrap Method

A 6-Point Example

Summary and Outlook

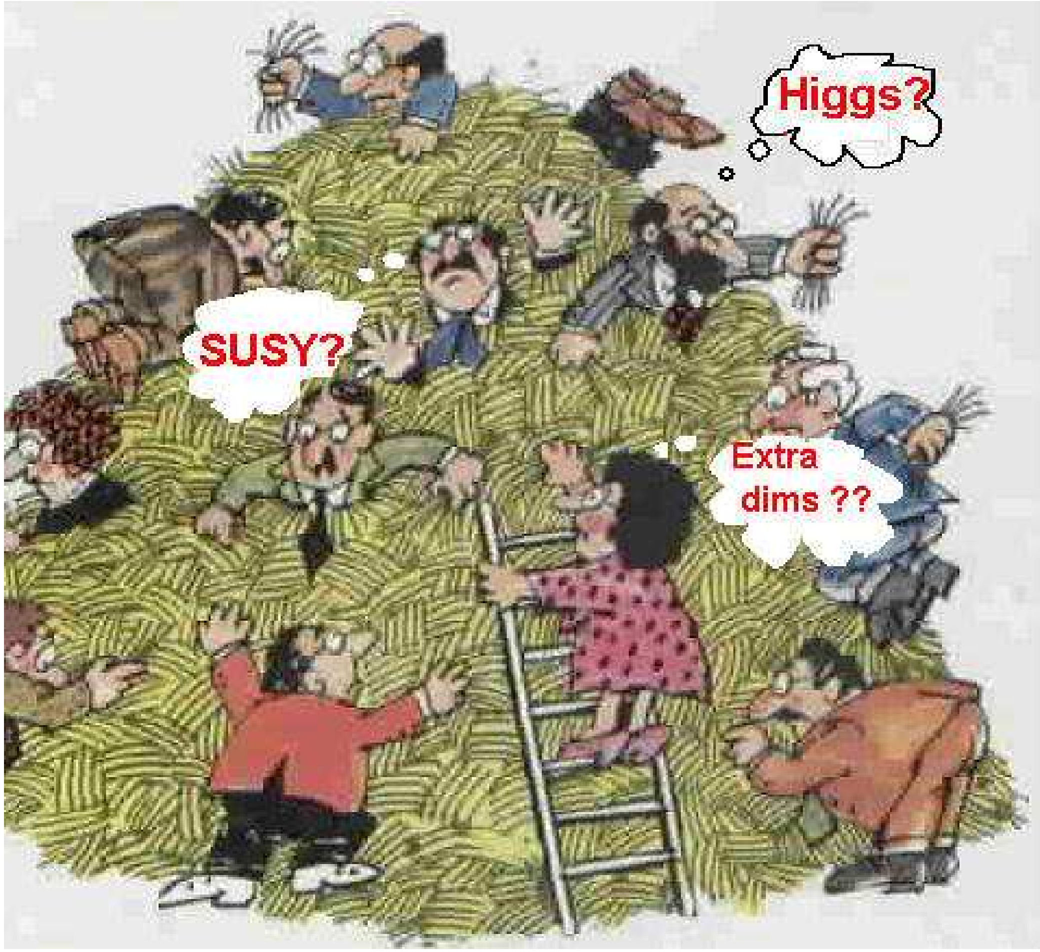
SUSY search - missing E_T+ jets

SM background from $W/Z(\to \nu\bar{\nu})+$ jets



Need to understand the "haystack" to find the needle!

Physics at the LHC . . .?



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Summary and Outlook

The (In)Famous Experimenters' Wishlists

Run II Monte Carlo Workshop 2001

Single boson	Diboson	Triboson	Heavy flavor
$W+\leq 5j$	$WW+\leq 5j$	$WWW+\leq 3j$	$t\bar{t}+\leq 3j$
$W+b\bar{b}+\leq 3j$	$WW+b\bar{b}+\leq 3j$	$WWW+b\bar{b}+\leq 3j$	$t\bar{t}+\gamma+\leq 2j$
$W+c\bar{c}+\leq 3j$	$WW+c\bar{c}+\leq 3j$	$WWW+\gamma\gamma+\leq 3j$	$t\bar{t}+W+\leq 2j$
$Z+\leq 5j$	$ZZ+\leq 5j$	$Z\gamma\gamma+\leq 3j$	$t\bar{t}+Z+\leq 2j$
$Z+b\bar{b}+\leq 3j$	$ZZ+b\bar{b}+\leq 3j$	$WZZ+\leq 3j$	$t\bar{t}+H+\leq 2j$
$Z+c\bar{c}+\leq 3j$	$ZZ+c\bar{c}+\leq 3j$	$ZZZ+\leq 3j$	$t\bar{b}+\leq 2j$
$\gamma+\leq 5j$	$\gamma\gamma+\leq 5j$		$tb\bar{b}+\leq 3j$
$\gamma+b\bar{b}+\leq 3j$	$\gamma\gamma+b\bar{b}+\leq 3j$		
$\gamma+c\bar{c}+\leq 3j$	$\gamma\gamma+c\bar{c}+\leq 3j$		
	$WZ+\leq 5j$		
	$WZ+b\bar{b}+\leq 3j$		
	$WZ+c\bar{c}+\leq 3j$		
	$W\gamma+\leq 3j$		
	$Z\gamma+\leq 3j$		

The (In)Famous Experimenters' Wishlists

Les Houches 2005

process wanted at NLO ($V \in \{Z, W, \gamma\}$)	background to
1. $pp \to VV +$ jet	$t\bar{t}H$, new physics
2. $pp \to H + 2$ jets	H production by vector boson fusion (VBF)
3. $pp \to t\bar{t}b\bar{b}$	$t\bar{t}H$
4. $pp \to t\bar{t} + 2$ jets	$t\bar{t}H$
5. $pp \to VVb\bar{b}$	VBF $\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to VV + 2$ jets	VBF $\to H \to VV$
7. $pp \to V + 3$ jets	new physics
8. $pp \to VVV$	SUSY trilepton

The (In)Famous Experimenters' Wishlists

Les Houches 2005

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5. $pp \to VVb\bar{b}$	VBF $\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to VV + 2$ jets	VBF $\to H \to VV$
7. $pp \to V + 3$ jets	new physics
8. $pp \to VVV$	SUSY trilepton

Large number of high-multiplicity processes that need to be computed!
The LHC turns on in 2007!

Precision Calculations

$$N = \mathcal{L} \sum_{i,j} \left(\int f_i(x_1) f_j(x_2) \sigma_{ij}(x_1, x_2) \right)$$

$$\sigma_{ij}(x_1, x_2) = \int d\text{PS}\, |\mathcal{M}_{ij}|^2$$

Need to know as precisely as possible:

- Luminosity
- PDFs
- Cross sections and branching fractions for signal and background
  - Amplitudes
  - Integration over final-state phase space – cancellation of IR divergences between real and virtual diagrams. Sometimes incomplete $\Rightarrow$ large logarithms $\Rightarrow$ resummation, parton showers (MC@NLO, . . .)

All of the above require lots of effort.

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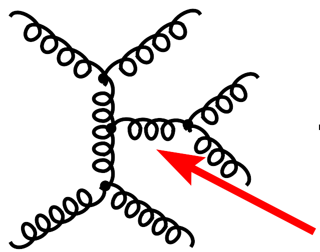
The Problem with Feynman Graphs

- Feynman rules are **too general, not optimized, do not take into account all symmetries of the theory**
- Vertices and propagators involve **gauge-dependent off-shell states**
- **Explosive growth** of number of diagrams/terms

gluon legs	tree level	one loop
6	220	1,034
8	34,300	3,017,490
∞	worse than ∞	even worse than ∞



+ 219 more

$p^2 \neq 0$

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Time to panic?? – No!

➡ **(Semi)Numerical approaches and automatization**

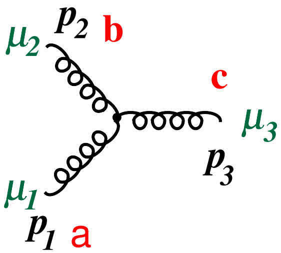
MadEvent, ALPGEN, CompHEP, GRACE, HELAC/PHEGAS, . . .

Campbell, Ellis, Giele, Glover, Zanderighi; Kramer, Soper, Nagy; Binoth, Ciccolini, Guillet, Heinrich, Kauer, Pilon, Schubert; Czakon; Anastasiou, Daleo; . . .

➡ **On-Shell recursion relations**

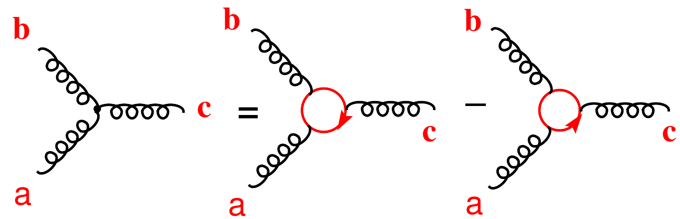
Simplifications for Calculating Amplitudes

Color structure



$$= ig\, f^{abc} \left[\eta_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + \eta_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + \eta_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}\right]$$

$$f^{abc} = -i \left[\mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c)\right]$$



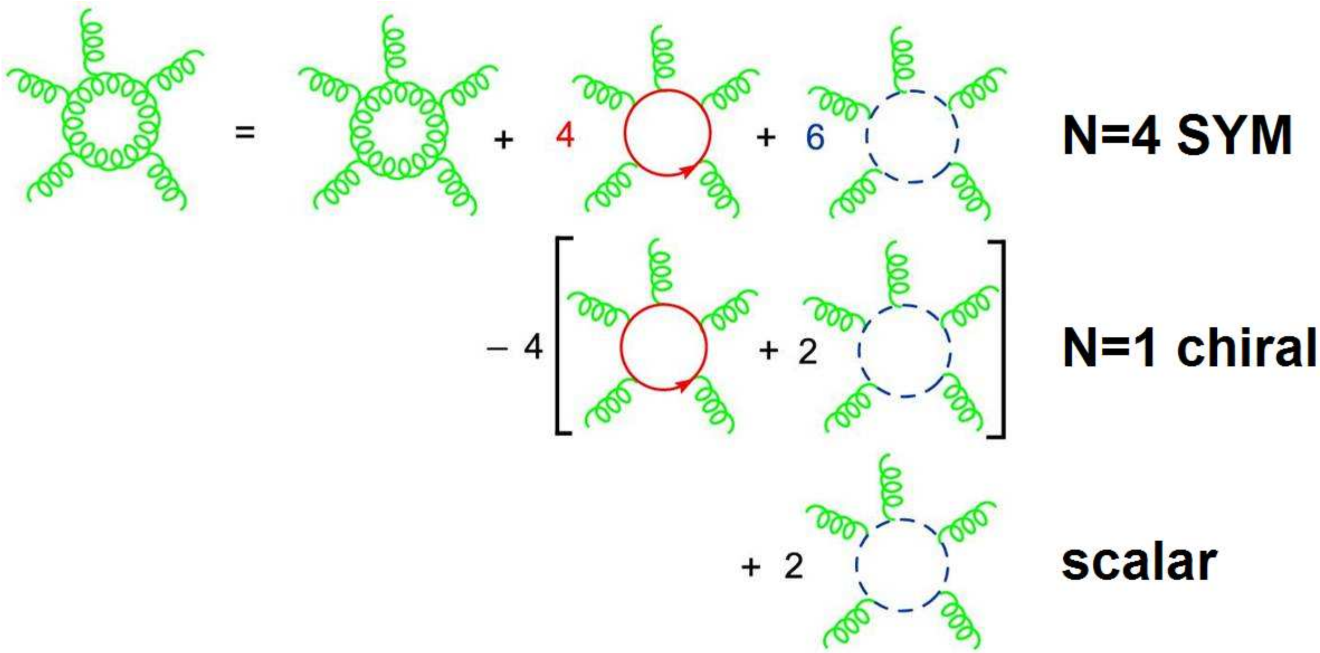
Strip color and coupling information, only calculate diagrams with cyclic color ordering.
All other diagrams can be obtained by permuting external legs. Set of n-gluon tree amplitudes:

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} .. T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1), .., \sigma(2))$$

Berends, Giele; Mangano, Parke, Xu; Bern, Kosower

Simplifications for Calculating Amplitudes

Supersymmetric decomposition of one-loop amplitudes



Bern, Dixon, Kosower

Simplifications for Calculating Amplitudes

Spinor helicity formalism

Use the "right variables" to expose more symmetries.
Instead of description of amplitudes in terms of momenta (spin 1) take "square root" (spin 1/2) – use Dirac spinors:

$$u_\pm(p) = |p^\pm\rangle = \frac{1}{2}(1 \pm \gamma_5)\, u(p)$$

Momentum invariants (spin 0) - antisymmetric product of spin 1/2:

$$(p_i + p_j)^2 = 2 p_i \cdot p_j = \langle i\, j\rangle\, [j\, i]$$

$$\begin{aligned}\langle i\, j\rangle &= \langle i^- | j^+\rangle = \bar{u}_-(p_i) u_+(p_j) \\ [i\, j] &= \langle i^+ | j^-\rangle = \bar{u}_+(p_i) u_-(p_j)\end{aligned}$$

Berends, Kleiss, De Causmaecker, Gastmans, Stirling, Troost, Wu; Xu, Zhang, Chang; Gunion, Kunszt

Recursion relations

"Recycle" known amplitudes – off-shell tree level recursions $\sim$ 20 years ago

Berends, Giele

One loop? Integrals?

Recursion relations

"Recycle" known amplitudes – off-shell tree level recursions $\sim$ 20 years ago

Berends, Giele

One loop? Integrals?

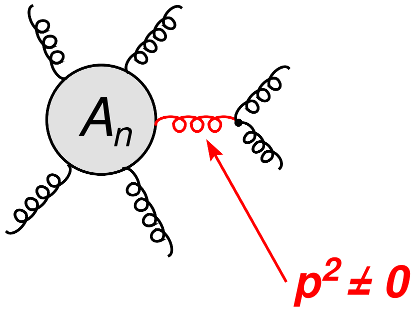
Transformation to Penrose's twistor space (Fourier transform in $u_-(p) = \frac{1}{2}(1-\gamma_5)u(p)$)
$\Rightarrow$ amazingly simple structure of scattering amplitudes

Parke, Taylor; Witten; Nair; Roiban, Spradlin, Volovich

$\Rightarrow$ On-shell recursion relations

Britto, Cachazo, Feng, Witten

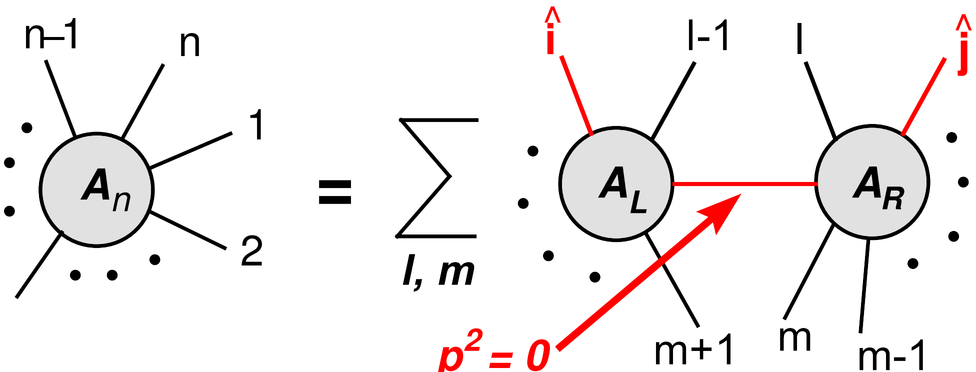
On-Shell Recursion Relations at Tree Level



Complex continue (shift) spinors and momenta:

$$p_i \to p_i(z) \qquad p_j \to p_j(z)$$
$$p_i + p_j \to p_i + p_j$$

Momentum conservation is maintained, momenta on-shell ($p_i(z)^2 = p_j(z)^2 = 0$).



Britto, Cachazo, Feng

Proof at Tree-Level

Propagators and thus amplitudes are now functions of the complex parameter:

$$1/P^2_{l\ldots j\ldots m} \quad \rightarrow \quad 1/P^2_{l\ldots j\ldots m}(z)$$

$$A(z) = \sum_{l,m} \sum_h A_L^h(z) \frac{1}{P^2_{l\ldots j\ldots m}(z)} A_R^{-h}(z)$$

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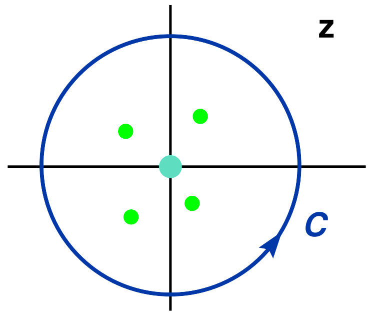
If $A(z \rightarrow \infty) \rightarrow 0$ **- Cauchy's theorem**



$$\frac{1}{2\pi i} \oint_C \frac{dz}{z} A(z) = 0$$

$$A(0) \quad = \quad - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

$$= \quad \sum_{\text{poles } \alpha} \sum_{h} A_L^h(z_\alpha) \frac{1}{P^2_{l\ldots j\ldots m}} A_R^{-h}(z_\alpha)$$

Britto, Cachazo, Feng, Witten

Massive Particles

No principal difference:

$$\frac{1}{P^2_{l\dots j\dots m} - M^2_{l\dots m}} \rightarrow \frac{1}{P^2_{l\dots j\dots m}(z) - M^2_{l\dots m}}$$

$$A(z) = \sum_{l,m} \sum_h A_L^h(z) \frac{1}{P^2_{l\dots j\dots m}(z) - M^2_{l\dots m}} A_R^{-h}(z)$$

Building blocks and residues more complicated, but same strategy.

Badger, Glover, Khoze, Svrcek

Applications at Tree Level

Proof at tree level only relies on Cauchy's theorem and basic factorization properties.

See also: Draggiotis, Kleiss, Lazopoulos, Papadopoulos; Vaman, Yao

$\Rightarrow$ **Many applications**

- **SUSY - processes with massless fermions** Luo, Wen
- **QCD - QCD is supersymmetric at tree level**
- **Massive scalars and fermions**

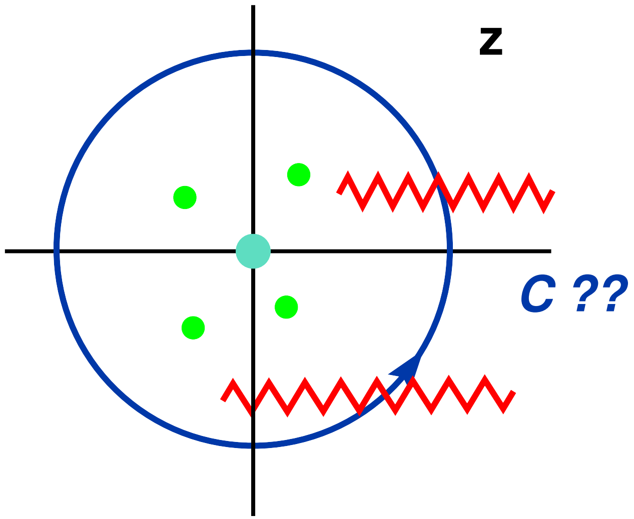
  Badger, Glover, Khoze, Svrcek; Forde, Kosower; Schwinn, Weinzierl; Ferrario, Rodrigo, Talavera
- **Higgs (top loop integrated out)** Badger, Dixon, Glover, Khoze
- **Gravity**

  Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager

"One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane."

in J. Schwinger, "Particles, Sources, and Fields", Vol. I.

QCD at One Loop - A Disaster?



- Branch cuts (with spurious singularities)

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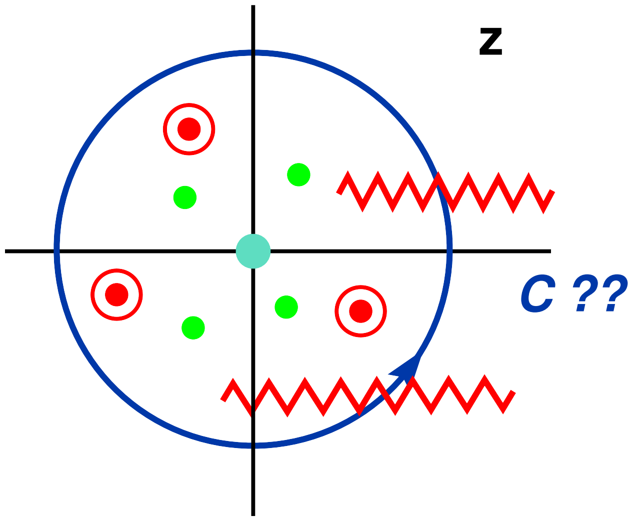
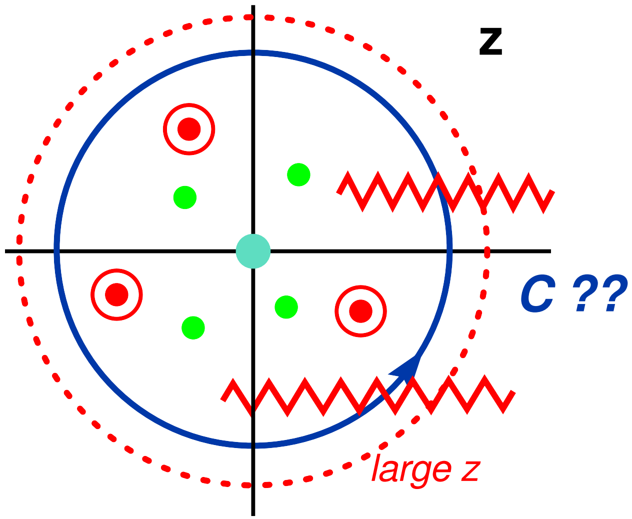


- **Branch cuts (with spurious singularities)**
- **Double poles, ‘unreal poles’ and nonstandard factorizations**

QCD at One Loop - A Disaster?



- Branch cuts (with spurious singularities)
- Double poles, ‘unreal poles’ and nonstandard factorizations
- $A(z \to \infty) \neq 0$

On-Shell Bootstrap Method

article | discussion | edit this page | history

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Bootstrap model

From Wikipedia, the free encyclopedia

In physics, the term **bootstrap model** is used for the class of theories that assume that very general consistency criteria are sufficient to determine the whole theory completely. In such theories, typically examples of quantum field theory, it is impossible to divide the objects and concepts to elementary and composite ones. See Geoffrey Chew. This strategy turned out to be successful only in the case of two-dimensional conformal field theory where many insights can indeed be derived by this method.

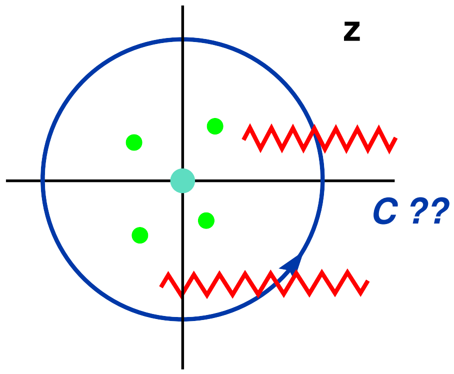
Here: very general consistency criteria

- **Cuts (unitarity)**
- **Poles (factorization)**

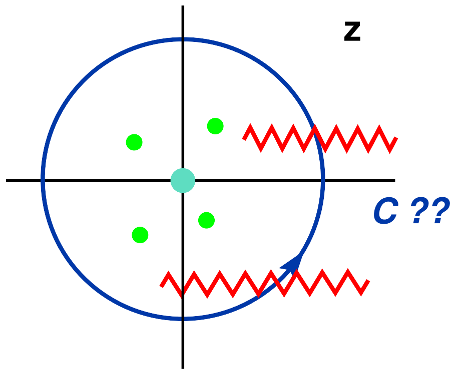
$$A(z) = C(z) + R(z)$$

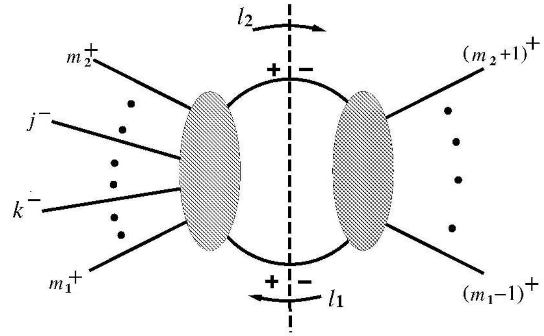
Factorize independently. But: C and R talk to each other via behavior at $z \to \infty$ and spurious singularities! Need to keep this in mind when constructing recursion relations for R.

Cut Parts



Cut Parts





$C(0)$ contains only Li, ln, π^2 – **cut-constructible! via (generalized) unitarity**

$$\int d\mathrm{LIPS}(-l_1, l_2) A^{\mathrm{tree}}(-l_1, m_1, .., m_2, l_2) A^{\mathrm{tree}}(-l_2, m_2{+}1, .., m_1{-}1, l_1)$$

Trees "recycled" into loops

Bern, Dixon, Dunbar, Kosower

Get tree graphs via on-shell recursion relations.
Integrate directly. Bedford, Brandhuber, McNamara, Spence, Travaglini

Cut Parts – Ongoing Developments

$$A = \sum_{i=1}^{4} (c_i + \sum_j S_{ij} b_i j) I_i + R$$

Coefficients obtainable by generalized unitarity – rewrite in terms of spinor-integration, express spinor-integration as total derivative such that integration is reduced to reading off residues at poles. Application of nonlinear Schouten identity to split up contributions such that each has only simple poles.

Britto, Buchbinder, Cachazo, Feng, Mastrolia

Alternative: rewrite numerator of integrand by suitable expansion in terms of convenient basis, successively subtract off $c_4, b_{4j}; c_3, b_{3j}; \ldots$ by solving systems of equations.

Ossola, Papadopoulos, Pittau; Binoth, Guillet, Heinrich

In some cases coefficients of amplitudes with more legs recursively constructible from coefficients of amplitudes with fewer legs.

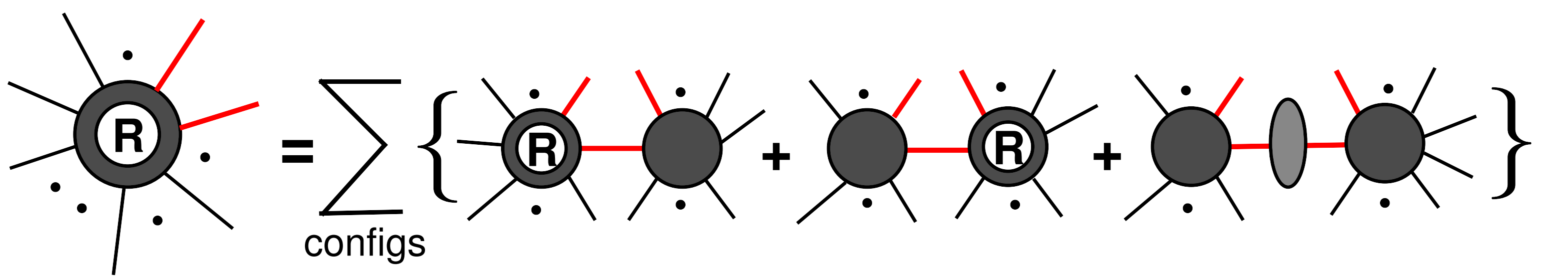
Bern, Bjerrum-Bohr, Dunbar, Ita

On-Shell Recursion for Rational Parts

$$A(z) = C(z) + R(z) \qquad \Bigg| \qquad \frac{1}{2\pi i}\oint_C \frac{dz}{z}$$

$$A(0) = C(0) + \text{Inf}\, A - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R(z)}{z}$$

$$= C(0) + \text{Inf}\, A + \sum_{\text{configs}} A_L \frac{1}{P^2_{l\ldots m}} A_R$$

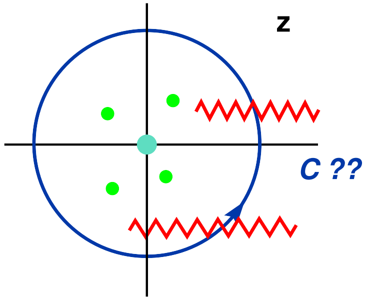


Loops "recycled" into loops

(ignoring slight subtleties with spurious singularities)

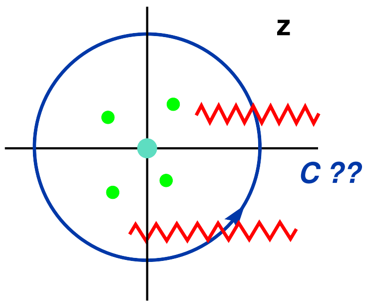
Bern, Dixon, Kosower

Non-Standard Factorizations



$$A(0) = C(0) \qquad - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R(z)}{z}$$

Non-Standard Factorizations



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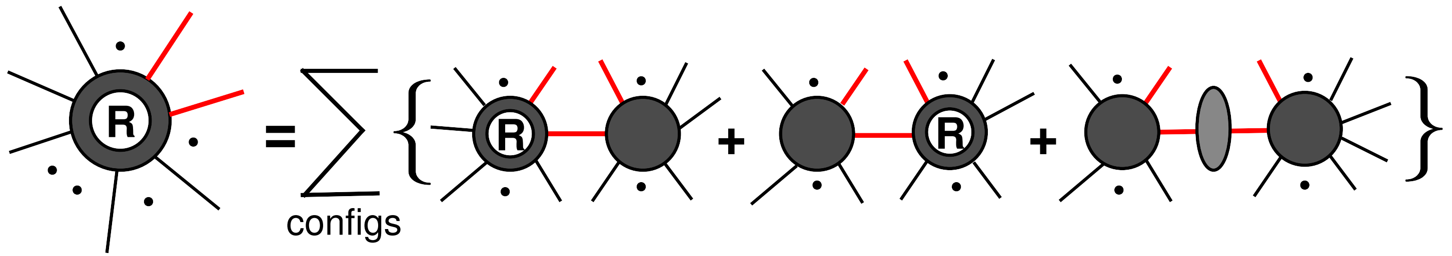
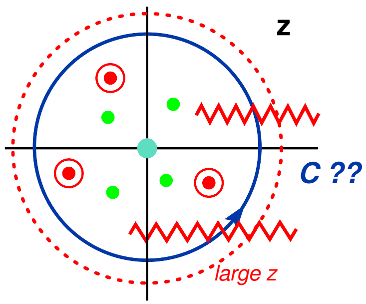


Non-Standard Factorizations



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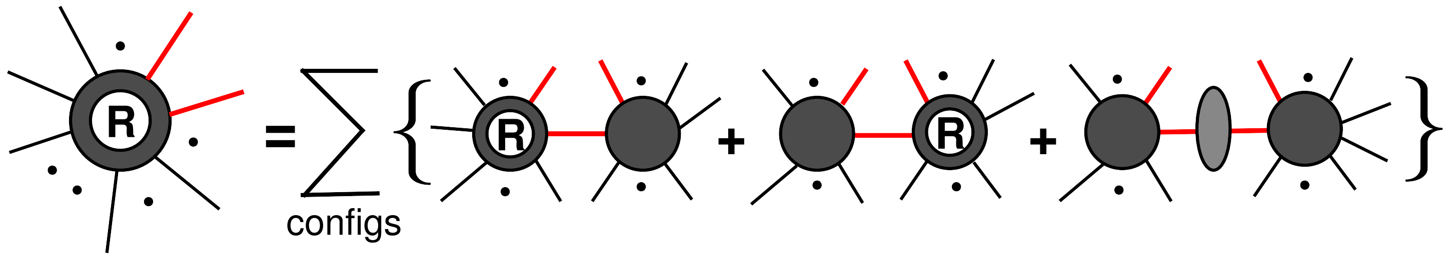
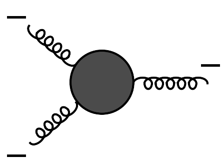
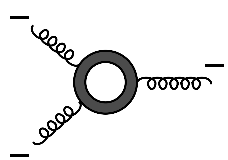
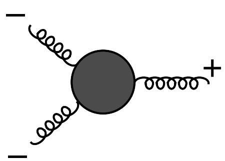


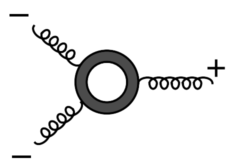


$= 0$



$\sim$ **double pole $+$ ratl** $\times(--+)$**tree**





$\sim$ **rational** $\times(--+)$**tree**

Factorization properties unclear at one loop.

Large-z Contributions

Can pick shifts to avoid either non-standard factorizations or $z \to \infty$ contributions, but in general not both!

- **Shift $[j, l\rangle$ avoids non-standard factorizations**

$$A(0) = C(0) + \operatorname*{Inf}_{[j,l\rangle} A + R^{[j,l\rangle}_{\text{recurs}}$$

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- **Shift $[j, l\rangle$ avoids non-standard factorizations**

$$A(0) = C(0) + \operatorname*{Inf}_{[j,l\rangle} A + R^{[j,l\rangle}_{\text{recurs}}$$

- **Shift $[a, b\rangle$ has no large-parameter contributions**

$$A(0) = C(0) + R^{[a,b\rangle}_{\text{recurs}} + \text{non-standard channels}^{[a,b\rangle}$$

The Bootstrap Formalism

Solution $\Rightarrow$ use two shifts!

Extract large-parameter contributions of primary shift from auxiliary relation:

$$A(0) = C(0) + R^{[a,b\rangle}_{\text{recurs}} + \text{non-standard}^{[a,b\rangle} \quad \Big| \; [j,l\rangle \; \Big| \; \operatorname*{Inf}_{[j,l\rangle}$$

$$\operatorname*{Inf}_{[j,l\rangle} A = \operatorname*{Inf}_{[j,l\rangle} C + \operatorname*{Inf}_{[j,l\rangle} R^{[a,b\rangle}_{\text{recurs}}$$

$$\text{if} \quad \operatorname*{Inf}_{[j,l\rangle} \left[\text{non-standard channels}^{[a,b\rangle}\right] = 0$$

The Bootstrap Formalism

Solution ⇒ use two shifts!

Extract large-parameter contributions of primary shift from auxiliary relation:

$$A(0) = C(0) + R^{[a,b\rangle}_{\text{recurs}} + \text{non-standard}^{[a,b\rangle} \quad \Big| \; [j,l\rangle \; \Big| \; \underset{[j,l\rangle}{\mathrm{Inf}}$$

$$\underset{[j,l\rangle}{\mathrm{Inf}}\, A = \underset{[j,l\rangle}{\mathrm{Inf}}\, C + \underset{[j,l\rangle}{\mathrm{Inf}}\, R^{[a,b\rangle}_{\text{recurs}}$$

$$\text{if} \quad \underset{[j,l\rangle}{\mathrm{Inf}} \left[\text{non-standard channels}^{[a,b\rangle}\right] = 0$$

The complete bootstrap

$$\boxed{A(0) = C(0) + R^{[j,l\rangle}_{\text{recurs}} + \underset{[j,l\rangle}{\mathrm{Inf}} \left[C + R^{[a,b\rangle}_{\text{recurs}} \right]}$$

Passes all nontrivial checks!

CFB, Bern, Dixon, Forde, Kosower

Example: $A^{(1)}(1^-,2^-,3^-,4^+,5^+,6^+)$

(1)
$$X(1,2,3,4,5,6)\Big|_{\text{flip 1}} \equiv X(3,2,1,6,5,4)$$

$$\widehat{C}_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{1}{3c_\Gamma}\,A_{6;1}^{\mathcal{N}=1}(1^-,2^-,3^-,4^+,5^+,6^+)$$

(2)
$$+\ \frac{2}{9}A_6^{\text{tree}}(1^-,2^-,3^-,4^+,5^+,6^+) + \hat{C}_6^a + \hat{C}_6^a\Big|_{\text{flip 1}}$$

$$\hat{C}_6^a =$$

$$\frac{i}{3}\left[\frac{\langle 1\,2\rangle\langle 2\,3\rangle[2\,4]\langle 1^-|\,(3+4)\,|2^-\rangle\Big[\langle 3^-|\,42\,|1^+\rangle P_{234}^2 - \langle 3^-|\,2(3+4)\,|1^+\rangle P_{34}^2\Big]}{\langle 3\,4\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[2\,3]\,\langle 5^-|\,(3+4)\,|2^-\rangle}\,\frac{\mathsf{L}_2(\frac{-P_{234}^2}{-P_{34}^2})}{(P_{34}^2)^3}\right.$$

$$\left. + \frac{\langle 3\,5\rangle[4\,5][5\,6]\langle 5^-|\,(1+2)\,|6^-\rangle\Big[\langle 3^-|\,(5-4)\,|6^-\rangle P_{345}^2 + \langle 3^-|\,(4+5)\,|6^-\rangle P_{34}^2\Big]}{\langle 4\,5\rangle\,[1\,2]\,[1\,6]\,\langle 5^-|\,(3+4)\,|2^-\rangle}\,\frac{\mathsf{L}_2(\frac{-P_{345}^2}{-P_{34}^2})}{(P_{34}^2)^3}\right]$$

$$\mathsf{L}_2(r) = \frac{\ln(r)-(r-1/r)/2}{(1-r)^3}$$

Bern, Bjerrum-Bohr, Dunbar, Ita

Example: $A^{(1)}(1^-,2^-,3^-,4^+,5^+,6^+)$

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$$\mathsf{L}_2(r) = \frac{\ln(r)-(r-1/r)/2}{(1-r)^3}$$

Bern, Bjerrum-Bohr, Dunbar, Ita

Shift $[1,2\rangle$

Example: $A^{(1)}(1^-,2^-,3^-,4^+,5^+,6^+)$ contd.

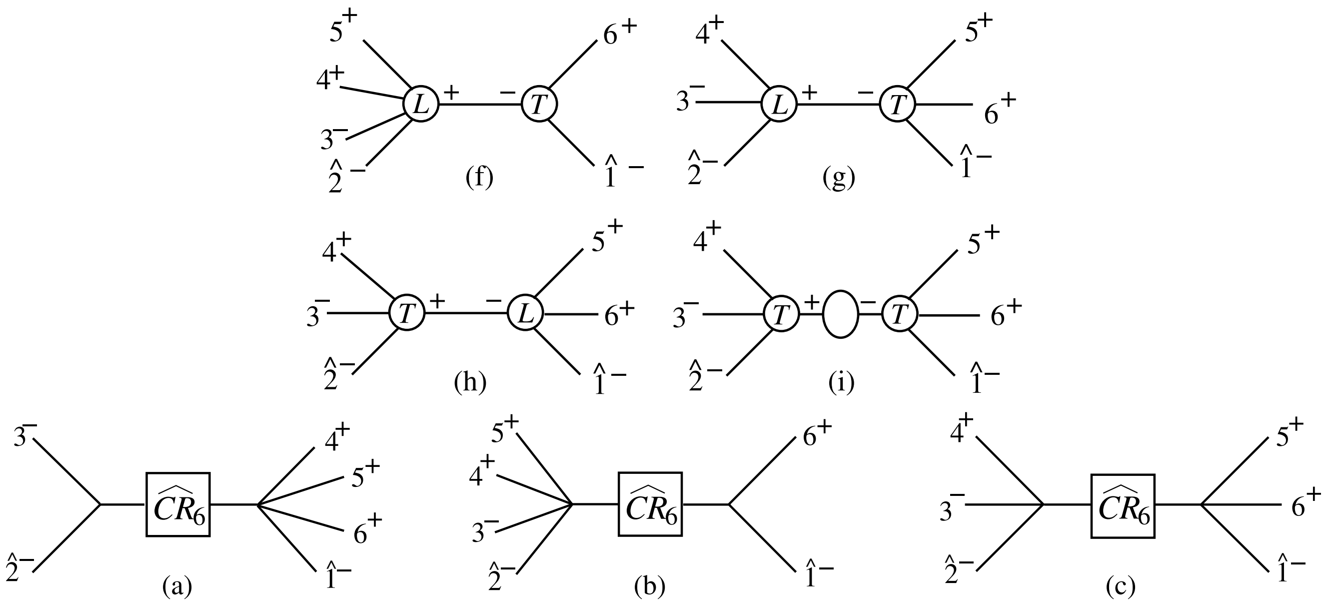
$$\Rightarrow \mathrm{Inf}_{[1,2\rangle}\, \widehat{C}_6 = \lim_{z\to\infty} \widehat{C}_6$$

Recursive and overlap contributions in channels

$$(3) \qquad P_{61}^2 \to P_{61}^2 - z\,\langle 1^- |\, \not{P}_{61}\, | 2^- \rangle$$

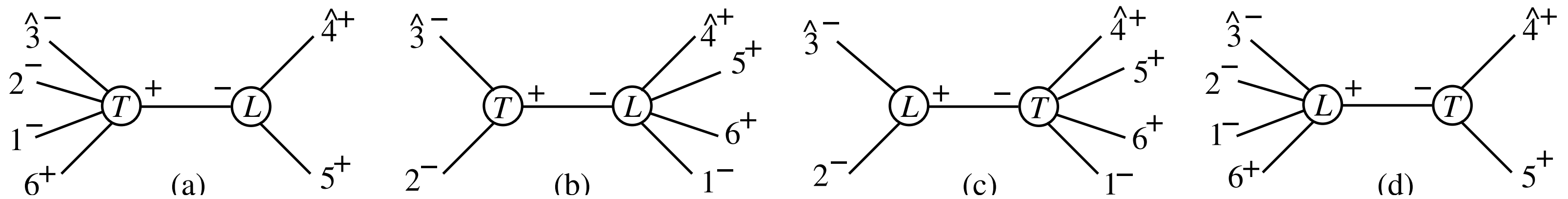
$$(4) \qquad P_{23}^2 \to P_{23}^2 + z\,\langle 1^- |\, \not{P}_{23}\, | 2^- \rangle$$

$$(5) \qquad P_{234}^2 \to P_{234}^2 + z\,\langle 1^- |\, \not{P}_{234}\, | 2^- \rangle$$



Example: $A^{(1)}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ contd.

Auxiliary recursion relation for $\mathrm{Inf}_{[1,2\rangle}\, A$



$$\mathop{\mathrm{Inf}}_{[1,2\rangle} A_{6;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) =$$

$$\mathop{\mathrm{Inf}}_{[1,2\rangle} A_{5;1}(1^-, 2^-, \hat{3}^-, \hat{K}^+_{45}, 6^+)\, \frac{i}{P^2_{45}}\, A_3^{\mathrm{tree}}(-\hat{K}^-_{45}, \hat{4}^+, 5^+) \quad (6)$$

Example: $A^{(1)}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ contd.

(7)
$$\widehat{R}_6 = \widehat{R}_6^a + \widehat{R}_6^a\Big|_{\text{flip } 1}$$

$$\widehat{R}_6^a = \frac{i}{6}\frac{1}{[2\,3]\,\langle 5\,6\rangle\,\langle 5^-|\,(3+4)\,|2^-\rangle}\Bigg\{-\frac{[4\,6]^3\,[2\,5]\,\langle 5\,6\rangle}{[1\,2]\,[3\,4]\,[6\,1]} - \frac{\langle 1\,3\rangle^3\,\langle 2\,5\rangle\,[2\,3]}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,1\rangle}$$
$$+\frac{\langle 1^-|\,(2+3)\,|4^-\rangle^2}{[3\,4]\,\langle 6\,1\rangle}\left(\frac{\langle 1^-|\,2\,|4^-\rangle - \langle 1^-|\,5\,|4^-\rangle}{P_{234}^2} + \frac{\langle 1\,3\rangle}{\langle 3\,4\rangle} - \frac{[4\,6]}{[6\,1]}\right)$$
$$-\frac{\langle 1\,3\rangle^2(3\,\langle 1^-|\,2\,|4^-\rangle + \langle 1^-|\,3\,|4^-\rangle)}{\langle 3\,4\rangle\,\langle 6\,1\rangle}$$
$$+\frac{[4\,6]^2(3\,\langle 1^-|\,5\,|4^-\rangle + \langle 1^-|\,6\,|4^-\rangle)}{[3\,4]\,[6\,1]}\Bigg\}$$

$\Rightarrow$ **all-n solution**

CFB, Bern, Dixon, Forde, Kosower

Results

- All-multiplicity formulae for $(+ + \ldots +)$, $(- + \ldots +)$ one-loop gluon amplitudes (also with a fermion pair)

  Bern, Dixon, Kosower

- All-multiplicity formulae for $(+ \ldots + - + \ldots + - + \ldots +)$ one-loop gluon amplitudes

  Forde, Kosower; CFB, Bern, Dixon, Forde, Kosower

- All-multiplicity formulae for $(- - - + \ldots +)$ one-loop gluon amplitudes

  CFB, Bern, Dixon, Forde, Kosower

- Some all-multiplicity results for parts of Higgs plus gluons (and fermion pair) at NNLO (effective theory - top loop integrated out)

  CFB, Del Duca, Dixon

All of the above $\ll \infty$ pages

- Working algorithm for all other configurations of one-loop gluon amplitudes!

  CFB, Bern, Dixon, Forde, Kosower

To-Do List

- ❑ Fermions
- ❑ Massive partons
  On-shell recursion for rational terms should be straightforward; need algorithm for cut parts that is suitable for bootstrap application.
- ❑ Automatization
- ❑ Attack the wishlists...
- ❑ Understand complex factorization at one loop and beyond
  Connection to Lagrangian?
- ❑ Higher loops?

Physics at the LHC