Bootstrapping One-Loop Amplitudes
Or: Needles and Large HaystaCks

Carola F. Berger
*Stanford Linear Accelerator Center*

in collaboration with
Zvi Bern, Lance Dixon, Darren Forde, David Kosower
and Vittorio Del Duca

Fermilab – Dec. 7th, 2006
Outline

- Introduction to precision calculations relevant for the LHC:
  Why we (= theorists) haven’t fulfilled the experimenters’ wishlist yet
- A new method: On-shell recursion relations
  But: QCD at one loop is not so simple...
- On-shell bootstrap at one loop
- Summary, open questions, and outlook
  The wishlist will get done
Particle Physics in the 20th Century

What the LEP, Tevatron, … told us

You were right: There's a needle in this haystack…
NLO Calculations are Needed!

SUSY search - missing $E_T + \text{jets}$

SM background from $W/Z(\rightarrow \nu \bar{\nu}) + \text{jets}$

Need to understand the “haystack” to find the needle!
Particle Physics in the 21st Century?

Physics at the LHC . . . ?

- Higgs?
- SUSY?
- Extra dims ??
- On-Shell Recursion Relations at Tree Level
- The Bootstrap Method
- A 6-Point Example
- Summary and Outlook
The (In)Famous Experimenters’ Wishlists

<table>
<thead>
<tr>
<th>Single boson</th>
<th>Diboson</th>
<th>Triboson</th>
<th>Heavy flavor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W + \leq 5j$</td>
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<td>(t\bar{t}H), new physics</td>
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Les Houches 2005

Carola F. Berger — Fermilab — Dec. 7th, 2006
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Large number of high-multiplicity processes that need to be computed! The LHC turns on in 2007!
Precision Calculations

\[ N = \mathcal{L} \sum_{i,j} \left( \int f_i(x_1) f_j(x_2) \sigma_{ij}(x_1, x_2) \right) \]

\[ \sigma_{ij}(x_1, x_2) = \int d\text{PS} |\mathcal{M}_{ij}|^2 \]

Need to know as precisely as possible:

- Luminosity
- PDFs
- Cross sections and branching fractions for signal and background
  - Amplitudes
  - Integration over final-state phase space – cancellation of IR divergences between real and virtual diagrams. Sometimes incomplete \( \Rightarrow \) large logarithms \( \Rightarrow \) resummation, parton showers (MC@NLO, \ldots)

All of the above require lots of effort.
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All of the above require lots of effort.
The Problem with Feynman Graphs

- Feynman rules are too general, not optimized, do not take into account all symmetries of the theory
- Vertices and propagators involve gauge-dependent off-shell states
- Explosive growth of number of diagrams/terms

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+ 219 more

$p^2 \neq 0$
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Time to panic?? – No!

(Semi)Numerical approaches and automatization

MadEvent, ALPGEN, CompHEP, GRACE, HELAC/PHEGAS, ...

Campbell, Ellis, Giele, Glover, Zanderighi; Kramer, Soper, Nagy; Binoth, Ciccolini, Guillet, Heinrich, Kauer, Pilon, Schubert; Czakon; Anastasiou, Daleo; ...

On-Shell recursion relations
Color structure

\[ \mu_1 \quad p_1 \quad a \quad \mu_2 \quad p_2 \quad b \quad \mu_3 \quad p_3 \]

\[ f^{abc} = i g f^{abc} \left[ \eta_{\mu_1 \mu_2} (p_1 - p_2) \mu_3 + \eta_{\mu_2 \mu_3} (p_2 - p_3) \mu_1 + \eta_{\mu_3 \mu_1} (p_3 - p_1) \mu_2 \right] \]

Strip color and coupling information, only calculate diagrams with cyclic color ordering.
All other diagrams can be obtained by permuting external legs. Set of \( n \)-gluon tree amplitudes:

\[ A^{\text{tree}}_n = g^{n-2} \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}) A^{\text{tree}}_n (\sigma(1), \ldots, \sigma(2)) \]

Berends, Giele; Mangano, Parke, Xu; Bern, Kosower
Supersymmetric decomposition of one-loop amplitudes

\[ \text{N=4 SYM} \]
\[ \text{N=1 chiral} \]
\[ \text{scalar} \]

\[ = \quad + \quad 4 \quad + \quad 6 \]
\[ - \quad 4 \quad + \quad 2 \quad + \quad 2 \]

Bern, Dixon, Kosower
Simplifications for Calculating Amplitudes

Spinor helicity formalism
Use the “right variables” to expose more symmetries.
Instead of description of amplitudes in terms of momenta (spin 1) take “square root” (spin 1/2) – use Dirac spinors:

\[ u_\pm(p) = |p^\pm \rangle = \frac{1}{2} (1 \pm \gamma_5) u(p) \]

Momentum invariants (spin 0) - antisymmetric product of spin 1/2:

\[ (p_i + p_j)^2 = 2p_i \cdot p_j = \langle i \ j \rangle [j \ i] \]
\[ \langle i \ j \rangle = \langle i^- | j^+ \rangle = \bar{u}_-(p_i) u_+(p_j) \]
\[ [i \ j] = \langle i^+ | j^- \rangle = \bar{u}_+(p_i) u_-(p_j) \]

Berends, Kleiss, De Causmaecker, Gastmans, Stirling, Troost, Wu; Xu, Zhang, Chang;

Gunion, Kunszt
Recursion relations

“Recycle” known amplitudes – off-shell tree level recursions ~ 20 years ago

One loop? Integrals?
Recursion relations

“Recycle” known amplitudes – off-shell tree level recursions ~ 20 years ago

Berends, Giele

One loop? Integrals?

Transformation to Penrose’s twistor space (Fourier transform in $u_-(p) = \frac{1}{2} (1 - \gamma_5) u(p)$)

⇒ amazingly simple structure of scattering amplitudes

Parke, Taylor; Witten; Nair; Roiban, Spradlin, Volovich

⇒ On-shell recursion relations

Britto, Cachazo, Feng, Witten
On-Shell Recursion Relations at Tree Level

Complex continue (shift) spinors and momenta:

\[ p_i \rightarrow p_i(z) \quad p_j \rightarrow p_j(z) \]

\[ p_i + p_j \rightarrow p_i + p_j \]

Momentum conservation is maintained, momenta on-shell \((p_i(z)^2 = p_j(z)^2 = 0)\).
Propagators and thus amplitudes are now functions of the complex parameter:

\[
\frac{1}{P_{l\ldots j\ldots m}^2} \rightarrow \frac{1}{P_{l\ldots j\ldots m}^2(z)}
\]

\[
A(z) = \sum_{l,m} \sum_h A_L^h(z) \frac{1}{P_{l\ldots j\ldots m}^2(z)} A_R^{-h}(z)
\]

Britto, Cachazo, Feng, Witten
Proof at Tree-Level

Propagators and thus amplitudes are now functions of the complex parameter:

\[ \frac{1}{P_{l...j...m}^2} \rightarrow \frac{1}{P_{l...j...m}(z)} \]

\[ A(z) = \sum_{l,m} \sum_{h} A^h_L(z) \frac{1}{P_{l...j...m}^2(z)} A^{-h}_R(z) \]

If \( A(z \rightarrow \infty) \rightarrow 0 \) - Cauchy’s theorem

\[ \frac{1}{2\pi i} \oint_C \frac{dz}{z} A(z) = 0 \]

\[ A(0) = - \sum_{\text{poles } \alpha} \text{Res}_{z=\alpha} \frac{A(z)}{z} \]

\[ = \sum_{\text{poles } \alpha} \sum_{h} A^h_L(\alpha) \frac{1}{P_{l...j...m}^2} A^{-h}_R(\alpha) \]

Britto, Cachazo, Feng, Witten
Massive Particles

No principal difference:

\[
\frac{1}{P_{l...j...m}^2 - M_{l...m}^2} \rightarrow \frac{1}{P_{l...j...m}(z)^2 - M_{l...m}^2}
\]

\[
A(z) = \sum_{l,m} \sum_h A^h_L(z) \frac{1}{P_{l...j...m}(z)^2 - M_{l...m}^2} A^h_R(z)
\]

Building blocks and residues more complicated, but same strategy.

Badger, Glover, Khoze, Svrcek
Proof at tree level only relies on Cauchy’s theorem and basic factorization properties.

See also: Draggiotis, Kleiss, Lazopoulos, Papadopoulos; Vaman, Yao

⇒ Many applications

- SUSY - processes with massless fermions
  Luo, Wen
- QCD - QCD is supersymmetric at tree level
- Massive scalars and fermions
  Badger, Glover, Khoze, Svrcek; Forde, Kosower; Schwinn, Weinzierl; Ferrario, Rodrigo, Talavera
- Higgs (top loop integrated out)
  Badger, Dixon, Glover, Khoze
- Gravity
  Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager
“One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane.”

in J. Schwinger, “Particles, Sources, and Fields”, Vol. I.
QCD at One Loop - A Disaster?

- Branch cuts (with spurious singularities)
QCD at One Loop - A Disaster?

- Branch cuts (with spurious singularities)
- Double poles, ‘unreal poles’ and nonstandard factorizations
QCD at One Loop - A Disaster?

- Branch cuts (with spurious singularities)
- Double poles, ‘unreal poles’ and nonstandard factorizations
- \( A(z \rightarrow \infty) \neq 0 \)
On-Shell Bootstrap Method

Here: very general consistency criteria

- **Cuts (unitarity)**
- **Poles (factorization)**

\[ A(z) = C(z) + R(z) \]

Factorize independently. **But:** \( C \) and \( R \) talk to each other via behavior at \( z \to \infty \) and spurious singularities! Need to keep this in mind when constructing recursion relations for \( R \).
Cut Parts

Get tree graphs via on-shell recursion relations. Integrate directly.
Cut Parts

\( C(0) \) contains only Li, Ln, \( \pi^2 \) – cut-constructible! via
(generalized) unitarity

\[ \int d\text{LIPS}(-l_1, l_2) A_{\text{tree}}(-l_1, m_1, .., m_2, l_2) A_{\text{tree}}(-l_2, m_2+1, .., m_1-1, l_1) \]

Trees “recycled” into loops

Get tree graphs via on-shell recursion relations. Integrate directly.

\[ \text{Bern, Dixon, Dunbar, Kosower} \]
\[ \text{Bedford, Brandhuber, McNamara, Spence, Travaglini} \]
\[ A = \sum_{i=1}^{4} (c_i + \sum_j S_{ij} b_{ij}) I_i + R \]

Coefficients obtainable by generalized unitarity – rewrite in terms of spinor-integration, express spinor-integration as total derivative such that integration is reduced to reading off residues at poles. Application of nonlinear Schouten identity to split up contributions such that each has only simple poles. Application of nonlinear Schouten identity to split up contributions such that each has only simple poles.

Alternative: rewrite numerator of integrand by suitable expansion in terms of convenient basis, successively subtract off \( c_4, b_{4j}; c_3, b_{3j}; \ldots \) by solving systems of equations.

In some cases coefficients of amplitudes with more legs recursively constructible from coefficients of amplitudes with fewer legs.
On-Shell Recursion for Rational Parts

\[ A(z) = C(z) + R(z) \quad \text{with} \quad \frac{1}{2\pi i} \oint_C \frac{dz}{z} \]

\[ A(0) = C(0) + \text{Inf} A - \sum_{\text{poles } \alpha} \text{Res}_{z = z_\alpha} \frac{R(z)}{z} \]

\[ = C(0) + \text{Inf} A + \sum_{\text{configs}} A_L \frac{1}{P^2_{l\ldots m}} A_R \]

Loops “recycled” into loops
(ignoring slight subtleties with spurious singularities)

Bern, Dixon, Kosower
Non-Standard Factorizations

\[ A(0) = C(0) \quad \text{and} \quad -\sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} R(z) \]

Factorization properties unclear at one loop.
Non-Standard Factorizations

\[ A(0) = C(0) \quad \text{and} \quad - \sum_{\text{poles } z = z_\alpha} \text{Res} \, \frac{R(z)}{z} \]

\[ = \sum \text{configs} \{ \begin{array}{ccc}
\text{R} & + & \text{R} \\
\text{R} & + & \text{R} \\
\end{array} \} \]

A 6-Point Example
Non-Standard Factorizations

\[ A(0) = C(0) + \text{Inf } A - \sum_{\text{poles } \alpha} \text{Res } \frac{R(z)}{z} \]

\[ = \sum_{\text{configs}} \left\{ \text{R} + \text{R} + \text{R} + \text{R} \right\} \]

\[ = 0 \]

\[ \sim \text{double pole} + \text{ratl} \times (-++ \text{tree}) \]

\[ \sim \text{rational} \times (-++) \text{tree} \]

Factorization properties unclear at one loop.
Large-z Contributions

Can pick shifts to avoid either non-standard factorizations or \( z \rightarrow \infty \) contributions, but in general not both!

- **Shift** \([j, l]\) avoids non-standard factorizations

\[
A(0) = C(0) + \text{Inf}_{[j,l]} A + R_{\text{recurs}}^{[j,l]}
\]
Can pick shifts to avoid either non-standard factorizations or \( z \to \infty \) contributions, but in general not both!

- **Shift** \([j, l]\) avoids non-standard factorizations

\[
A(0) = C(0) + \text{Inf} A + R_{\text{reurs}}^{[j, l]}
\]

- **Shift** \([a, b]\) has no large-parameter contributions

\[
A(0) = C(0) + R_{\text{reurs}}^{[a, b]} + \text{non-standard channels}^{[a, b]}
\]
The Bootstrap Formalism

Solution \( \Rightarrow \) use two shifts!
Extract large-parameter contributions of primary shift from auxiliary relation:

\[
A(0) = C(0) + R^{[a,b]}_{\text{recurs}} + \text{non-standard}^{[a,b]}_{\text{[j,l]}} + \text{Inf}^{[j,l]}
\]

\[
\text{Inf}^{[j,l]} A = \text{Inf}^{[j,l]} C + \text{Inf}^{[j,l]} R^{[a,b]}_{\text{recurs}}
\]

\[
\text{if} \quad \text{Inf}^{[j,l]} \text{[non-standard channels}^{[a,b]}_{\text{[j,l]}} \] = 0
The Bootstrap Formalism

Solution ⇒ use two shifts!
Extract large-parameter contributions of primary shift from auxiliary relation:

\[ A(0) = C(0) + R_{\text{recurs}}^{[a,b]} + \text{non-standard}^{[a,b]} \mid [j, l] \mid \text{Inf}^{[j,l]} \]

\[
\text{Inf} A = \text{Inf} C + \text{Inf} R_{\text{recurs}}^{[a,b]} \\
\text{if} \quad \text{Inf} [\text{non-standard channels}^{[a,b]}] = 0
\]

The complete bootstrap

\[
A(0) = C(0) + R_{\text{recurs}}^{[j,l]} + \text{Inf}^{[j,l]} \left[ C + R_{\text{recurs}}^{[a,b]} \right]
\]

Passes all nontrivial checks!

CFB, Bern, Dixon, Forde, Kosower
Example: \( A^{(1)}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \)

\[
X(1, 2, 3, 4, 5, 6) \bigg|_{\text{flip 1}} \equiv X(3, 2, 1, 6, 5, 4)
\]

\[
\hat{C}_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = \frac{1}{3c_\Gamma} A_{6;1}^{N=1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)
\]

\[
= \frac{2}{9} A_6^{\text{tree}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) + \hat{C}_6^a + \hat{C}_6^a \bigg|_{\text{flip 1}}
\]

\[
\hat{C}_6^a = \frac{i}{3} \left[ \frac{(12)(23)[24]}{\langle 34 \rangle \langle 56 \rangle \langle 61 \rangle [23]} \langle 5^- | (3+4) | 2^- \rangle \right]
\]

\[
\left[ \langle 3^- | 42 | 1^+ \rangle P_{234}^2 - \langle 3^- | 2(3+4) | 1^+ \rangle P_{34}^2 \right] \frac{L_2 \left( \frac{-P_{34}^2}{P_{234}^2} \right)}{(P_{34}^2)^3}
\]

\[
+ \frac{(35)[45][56]}{\langle 45 \rangle [12][16]} \langle 5^- | (3+4) | 2^- \rangle \left[ \langle 3^- | (5-4) | 6^- \rangle P_{345}^2 + \langle 3^- | (4+5) | 6^- \rangle P_{34}^2 \right] \frac{L_2 \left( \frac{-P_{345}^2}{P_{34}^2} \right)}{(P_{34}^2)^3}
\]

\[
L_2(r) = \frac{\ln(r) - (r - 1/r)/2}{(1-r)^3}/2
\]

\[\text{Bern, Bjerrum-Bohr, Dunbar, Ita}\]
Example: $A^{(1)}(1^{-}, 2^{-}, 3^{-}, 4^{+}, 5^{+}, 6^{+})$

\[ X(1, 2, 3, 4, 5, 6) \left|_{\text{flip 1}} \right. = X(3, 2, 1, 6, 5, 4) \]

\[ \hat{C}_6(1^{-}, 2^{-}, 3^{-}, 4^{+}, 5^{+}, 6^{+}) = \frac{1}{3c_r} A^N_{6;1} (1^{-}, 2^{-}, 3^{-}, 4^{+}, 5^{+}, 6^{+}) \]

\[ \left. + \frac{2}{9} A^\text{tree} (1^{-}, 2^{-}, 3^{-}, 4^{+}, 5^{+}, 6^{+}) + \hat{C}_6^a + \hat{C}_6^a \right|_{\text{flip 1}} \]

\[ \hat{C}_6^a = \]

\[ \left[ \frac{\langle 1 2 \rangle \langle 2 3 \rangle [2 4] \langle 1^{-} \mid (3 + 4) \mid 2^{-}\rangle \left[ \langle 3^{-} \mid 42 \mid 1^{+}\rangle P_{234}^2 - \langle 3^{-} \mid 2(3 + 4) \mid 1^{+}\rangle P_{34}^2 \right] L_2 \left( \frac{-P_{234}^2}{P_{34}^2} \right)}{(P_{34}^2)^3} \right. \]

\[ \left. \langle 3 4 \rangle \langle 5 6 \rangle \langle 6 1 \rangle [2 3] \langle 5^{-} \mid (3 + 4) \mid 2^{-}\rangle \right] \]

\[ + \left. \frac{\langle 3 5 \rangle [4 5] [5 6] \langle 5^{-} \mid (1 + 2) \mid 6^{-}\rangle \left[ \langle 3^{-} \mid (5 - 4) \mid 6^{-}\rangle P_{345}^2 + \langle 3^{-} \mid (4 + 5) \mid 6^{-}\rangle P_{34}^2 \right] L_2 \left( \frac{-P_{345}^2}{P_{34}^2} \right)}{(P_{34}^2)^3} \right. \]

\[ \langle 4 5 \rangle \langle 1 2 \rangle \langle 1 6 \rangle \langle 5^{-} \mid (3 + 4) \mid 2^{-}\rangle \]

\[ L_2(r) = \frac{\ln(r) - (r - 1/r)/2}{(1-r)^3} \]

Bern, Bjerrum-Bohr, Dunbar, Ita

Shift [1, 2]
Example: $A^{(1)}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ contd.

$$\Rightarrow \text{Inf}_{[1,2]} \hat{C}_6 = \lim_{z \to \infty} \hat{C}_6$$

Recursive and overlap contributions in channels

(3) $P_{61}^2 \rightarrow P_{61}^2 - z \langle 1^- | P_{61} | 2^- \rangle$

(4) $P_{23}^2 \rightarrow P_{23}^2 + z \langle 1^- | P_{23} | 2^- \rangle$

(5) $P_{234}^2 \rightarrow P_{234}^2 + z \langle 1^- | P_{234} | 2^- \rangle$
Example: $A^{(1)}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ contd.

Auxiliary recursion relation for $\text{Inf}_{[1,2]} A$

$$\text{Inf}_{[1,2]} A_{6;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) =$$

$$\text{Inf}_{[1,2]} A_{5;1}(1^-, 2^-, 3^-, \hat{K}_{45}^+, 6^+) \frac{i}{P_{45}^2} A_{\text{tree}}^{\text{tree}}(-\hat{K}_{45}^-, \hat{4}^+, 5^+)$$

(6)
Example: $A^{(1)}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \text{ contd.}$

\[ \hat{R}_6 = \hat{R}_6^a + \hat{R}_6^a \big|_{\text{flip 1}} \]

\[ \hat{R}_6^a = \frac{i}{6} \frac{1}{[2 \ 3] \langle 5 \ 6 \rangle \langle 5^- | (3 + 4) | 2^- \rangle} \left\{ -\frac{[4 \ 6]^3 [2 \ 5] \langle 5 \ 6 \rangle}{[1 \ 2] [3 \ 4] [6 \ 1]} - \frac{\langle 1 \ 3 \rangle^3 [2 \ 5] [2 \ 3]}{\langle 3 \ 4 \rangle \langle 4 \ 5 \rangle \langle 6 \ 1 \rangle} \right. \\
\quad + \frac{\langle 1^- | (2 + 3) | 4^- \rangle^2}{[3 \ 4] \langle 6 \ 1 \rangle} \left( \frac{\langle 1^- \ 2 | 4^- \rangle - \langle 1^- \ 5 | 4^- \rangle}{P_{234}^2} + \frac{\langle 1 \ 3 \rangle}{\langle 3 \ 4 \rangle} - \frac{[4 \ 6]}{[6 \ 1]} \right) \right. \\
\quad - \frac{\langle 1 \ 3 \rangle^2 (3 \langle 1^- \ 2 | 4^- \rangle + \langle 1^- \ 3 | 4^- \rangle)}{\langle 3 \ 4 \rangle \langle 6 \ 1 \rangle} \\
\quad + \frac{[4 \ 6]^2 (3 \langle 1^- \ 5 | 4^- \rangle + \langle 1^- \ 6 | 4^- \rangle)}{[3 \ 4] [6 \ 1]} \left\} \]

⇒ all-n solution

CFB, Bern, Dixon, Forde, Kosower
Results

☑ All-multiplicity formulae for \((++\ldots+)\), \((-+\ldots+)\) one-loop gluon amplitudes (also with a fermion pair)
   
   Bern, Dixon, Kosower

☑ All-multiplicity formulae for 
\((+\ldots+-+\ldots+-+\ldots+)\) one-loop gluon amplitudes
   
   Forde, Kosower; CFB, Bern, Dixon, Forde, Kosower

☑ All-multiplicity formulae for \((-+-\ldots+)\) one-loop gluon amplitudes
   
   CFB, Bern, Dixon, Forde, Kosower

☑ Some all-multiplicity results for parts of Higgs plus gluons (and fermion pair) at NNLO (effective theory - top loop integrated out)
   
   CFB, Del Duca, Dixon

All of the above \(\ll \infty\) pages

☑ Working algorithm for all other configurations of one-loop gluon amplitudes!
   
   CFB, Bern, Dixon, Forde, Kosower
To-Do List

- Fermions
- Massive partons
  On-shell recursion for rational terms should be straightforward; need algorithm for cut parts that is suitable for bootstrap application.
- Automatization
- Attack the wishlists...
- Understand complex factorization at one loop and beyond
  Connection to Lagrangian?
- Higher loops?
Physics at the LHC

On-Shell Recursion
Relations at Tree Level

The Bootstrap Method

A 6-Point Example

Summary and Outlook

- Results
- To-Do List
- Physics at the LHC