

Physics of the D=5 Chern-Simons Term

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I. QED in $D=5$

- (a) Photons propagate in bulk; Naturalness, $m_e \ll 1/R$, requires chiral delocalization; Wilson-line mass term
- (b) Chiral delocalization requires a Chern-Simons term
- (c) Anomaly-freedom implies quantized coefficient of CS term; “consistent anomalies” vs. “covariant anomalies”
- (d) Chern-Simons term implies new interactions amongst bulk KK-mode photons, effective $D=4$ interaction,
 - (i) Use “Wilson-Line Gauge Transformation” to $A_5 = 0$
 - (ii) Large m_e limit \rightarrow Fermionic Dirac determinant modifies effective interaction; maintains gauge invariance
 - (iii) Compute $K' \rightarrow K + \text{gamma}$ via effective interaction

II. Yang-Mills in D=5

- (a) “quarks” on branes; gauge theory of flavor
compactify with A_5 zero-mode \rightarrow mesons $f_{\text{pi}} = 1/R$;
Wilson line mass term \leftrightarrow chiral condensate
- (b) Chiral delocalization requires a Chern-Simons term;
anomaly matching, quantization
- (c) Chern-Simons term implies bulk and
holographic interactions amongst KK-modes
effective D=4 interaction,
- (d) Large m_q limit \rightarrow Fermionic Dirac determinant modifies
effective interaction; maintains gauge invariance
- (e) Obtain effective interaction: **holographic part is
the full Wess-Zumino-Witten term.**

ANOMALIES, CHERN-SIMONS TERMS AND
CHIRAL DELOCALIZATION IN EXTRA DIMENSIONS.
FERMILAB-PUB-06-010-T (Jan 2006) 46p. [HEP-TH 0601154]

(to appear in Phys. Rev. D)

LECTURE NOTES FOR
MASSLESS SPINOR AND MASSIVE SPINOR
TRIANGLE DIAGRAMS.
FERMILAB-TM-2341-T (Jan 2006) 17p. [HEP-TH 0601155]

EXACT EQUIVALENCE OF THE D=4 GAUGED
WESS-ZUMINO-WITTEN TERM AND THE D=5
YANG-MILLS CHERN-SIMONS TERM.
FERMILAB-PUB-06-046-T (Mar 2006) 25p. [HEP-TH 0603060]

QED Chern-Simons term

D=3: Knot Theory \longleftrightarrow “Gauss’ Linking Theorem”

$$L_{CS} = \epsilon_{ijk} A^i \partial^j A^k$$

Bulk Physics: Photon Mass Term

Deser, Jackiw, Templeton, Schonfeld,
Siegel; Niemi, Semenoff, Y.S. Wu

D=5:
$$L_{CS} = \epsilon_{ABCDE} A^A \partial^B A^C \partial^D A^E$$

Bulk Physics: New interactions amongst KK-modes

Topological object: “instantonic soliton”

Deser's Theorem
Ramond and CTH

Associated Conserved Topological Currents:

$$\text{Singlet: } J_A = \epsilon_{ABCDE} \text{Tr}(G^{BC} G^{DE})$$

$$\text{Adjoint: } J_A^a = \epsilon_{ABCDE} \text{Tr}\left(\frac{\lambda^a}{2} \{G^{BC}, G^{DE}\}\right)$$

These currents come from a “completion” of the Lagrangian

Adjoint current - 2nd Chern character:

$$c\epsilon^{ABCDE} \text{Tr}\left(A_A \partial_B A_C \partial_D A_E - \frac{3i}{2} A_A A_B A_C \partial_D A_E - \frac{3}{5} A_A A_B A_C A_D A_E\right)$$

Singlet currents - auxiliary characters:

$$c' \epsilon_{ABCDE} V^A \text{Tr}(G^{BC} G^{DE})$$

The topology of the D=5 pure Yang Mills theory can be directly matched to the D=4

Chiral Lagrangian theory obtainable via deconstruction

Bianchi ID's, etc.:

CTH, CTH & Zachos

Mathematically exact matchings:

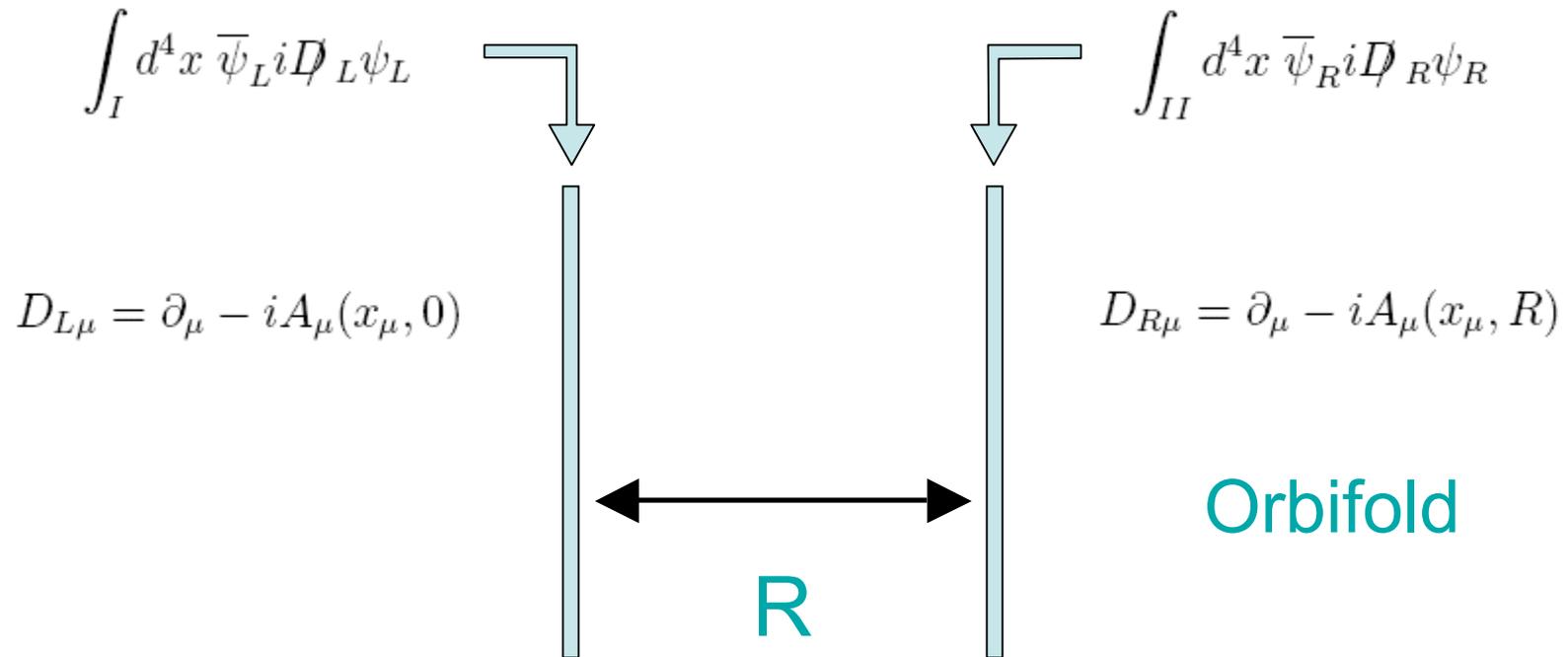
Instantonic Soliton \longleftrightarrow Skyrmion

Gauge currents \longleftrightarrow Chiral currents

Chern-Simons term
+ boundary term \longleftrightarrow WZW term

I. Technically natural QED in D=5:

Bulk: $D_A = \partial_A - iA_A$, $F_{AB} = i[D_A, D_B]$, $L_0 = -\frac{1}{4\tilde{e}^2}F_{AB}F^{AB}$



$m\bar{\psi}_L(x_\mu, 0)W\psi_R(x_\mu, R) + h.c.$

$W = \exp(i \int_0^R A_5(x_\mu, x_5) dx^5)$

Orbifold Boundary Conditions:

- (a) Horava-Witten
- (b) Magnetic Josephson Junction

Spectrum: (a) A_μ zero mode and KK tower
(b) No A_5 zero mode
(c) All A_5 modes eaten \rightarrow longitudinal dof's

Flipped Orbifold Boundary Conditions:

- (a) parity reversed Horava-Witten
- (b) Josephson Junction

Spectrum: (a) A_5 zero mode
(b) No A_μ zero mode
(c) All other A_5 modes eaten \rightarrow longitudinal dof's

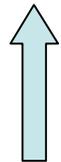
Gauge transformation in D=5:

$$A_A(x_\mu, y) \rightarrow A_A(x_\mu, y) + \partial_A \theta(x_\mu, y)$$

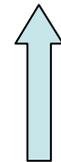
$$\psi_L(x_\mu) \rightarrow \exp(i\theta(0, x_\mu))\psi_L(x_\mu) \quad \psi_R(x_\mu) \rightarrow \exp(i\theta(R, x_\mu))\psi_R(x_\mu)$$

$$\begin{aligned} S_{branes} &\rightarrow S_{branes} + \int_I d^4x \bar{\psi}_L \gamma_\mu \partial^\mu \theta \psi_L(x_\mu, 0) + \int_{II} d^4x \bar{\psi}_R \gamma_\mu \partial^\mu \theta \psi_R(x_\mu, R) \\ &\rightarrow S_{branes} - \int_I d^4x \theta(x_\mu, 0) \partial_\mu J_L^\mu - \int_{II} d^4x \theta(x_\mu, R) \partial_\mu J_R^\mu \end{aligned}$$

$$\partial_\mu J_L^\mu = -\frac{1}{48\pi^2} F^{\mu\nu}(0) \tilde{F}_{\mu\nu}(0) \quad \partial_\mu J_R^\mu = \frac{1}{48\pi^2} F^{\mu\nu}(R) \tilde{F}_{\mu\nu}(R)$$

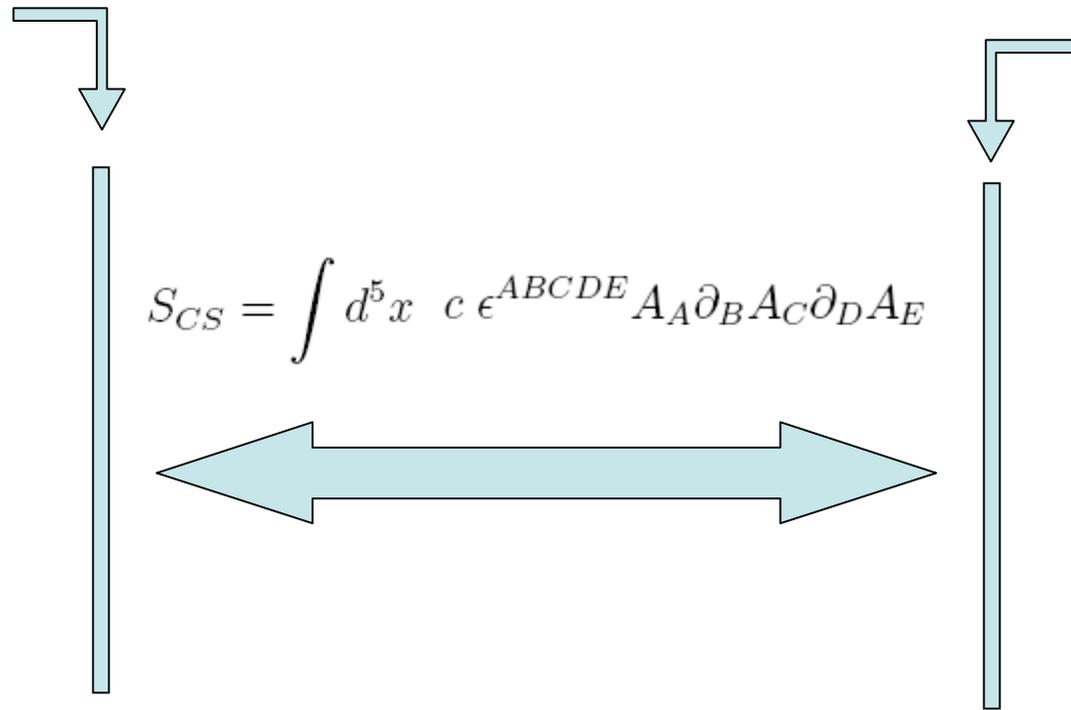


Consistent Anomalies



QED in D=5 requires Chern-Simons term:

$$L_{CS} = c \epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E = \frac{c}{4} \epsilon^{ABCDE} A_A F_{BC} F_{DE}$$



$$S_{CS} \rightarrow S_{CS} + \frac{c}{4} \int_{II} d^4x \ \theta(R) \ \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(R) - \frac{c}{4} \int_I d^4x \ \theta(0) \ \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(0) .$$

Anomaly Cancellation Condition:

$$S_{branes} \rightarrow S_{branes} + \frac{1}{48\pi^2} \int_I d^4x \theta(x_\mu, 0) F^{\mu\nu} \tilde{F}_{\mu\nu}(0) - \frac{1}{48\pi^2} \int_{II} d^4x \theta(x_\mu, R) F^{\mu\nu} \tilde{F}_{\mu\nu}(R)$$

$$S_{CS} \rightarrow S_{CS} - \frac{c}{2} \int_I d^4x \theta(x_\mu, 0) F^{\mu\nu} \tilde{F}_{\mu\nu} + \frac{c}{2} \int_{II} d^4x \theta(x_\mu, R) F^{\mu\nu} \tilde{F}_{\mu\nu}$$

Consistent Anomalies:

$$c = \frac{1}{24\pi^2}$$

Summary of Anomalies:

W. A. Bardeen, PR 184, 1848 (199)

Consistent Anomalies:

Consistent $L = V - A$ and $R = V + A$ Forms:

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j \gg m^2$):

$$\begin{aligned}\partial^\mu \bar{\psi} \gamma_\mu \psi_L &= -\frac{1}{48\pi^2} F_{L\mu\nu} \tilde{F}_L^{\mu\nu} \\ \partial^\mu \bar{\psi} \gamma_\mu \psi_R &= \frac{1}{48\pi^2} F_{R\mu\nu} \tilde{F}_R^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\partial^\mu \bar{\psi} \gamma_\mu \psi &= \frac{1}{12\pi^2} F_{V\mu\nu} \tilde{F}_A^{\mu\nu} \\ \partial^\mu \bar{\psi} \gamma_\mu \gamma^5 \psi &= \frac{1}{24\pi^2} (F_{V\mu\nu} \tilde{F}_V^{\mu\nu} + F_{A\mu\nu} \tilde{F}_A^{\mu\nu})\end{aligned}$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j \ll m^2$):

$$\begin{aligned}\partial^\mu \bar{\psi} \gamma_\mu \psi_L + im(\bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L) &= -\frac{1}{48\pi^2} F_{L\mu\nu} \tilde{F}_L^{\mu\nu} \\ \partial^\mu \bar{\psi} \gamma_\mu \psi_R + im(\bar{\psi}_R \psi_L - \bar{\psi}_L \psi_R) &= \frac{1}{48\pi^2} F_{R\mu\nu} \tilde{F}_R^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\partial^\mu \bar{\psi} \gamma_\mu \psi &= \frac{1}{12\pi^2} F_{V\mu\nu} \tilde{F}_A^{\mu\nu} \\ \partial^\mu \bar{\psi} \gamma_\mu \gamma^5 \psi - 2im\bar{\psi} \gamma^5 \psi &= \frac{1}{24\pi^2} (F_{V\mu\nu} \tilde{F}_V^{\mu\nu} + F_{A\mu\nu} \tilde{F}_A^{\mu\nu})\end{aligned}$$

$$\begin{aligned}\partial^\mu \bar{\psi} \gamma_\mu \psi_L &= \frac{1}{48\pi^2} (F_{L\mu\nu} \tilde{F}_R^{\mu\nu} + F_{R\mu\nu} \tilde{F}_L^{\mu\nu}) \\ \partial^\mu \bar{\psi} \gamma_\mu \psi_R &= -\frac{1}{48\pi^2} (F_{L\mu\nu} \tilde{F}_R^{\mu\nu} + F_{L\mu\nu} \tilde{F}_L^{\mu\nu})\end{aligned}$$

$$\begin{aligned}\partial^\mu \bar{\psi} \gamma_\mu \psi &= \frac{1}{12\pi^2} F_{V\mu\nu} \tilde{F}_A^{\mu\nu} \\ \partial^\mu \bar{\psi} \gamma_\mu \gamma^5 \psi &= -\frac{1}{12\pi^2} (F_{V\mu\nu} \tilde{F}_V^{\mu\nu})\end{aligned}$$

$$im\bar{\psi} \gamma^5 \psi \rightarrow -\frac{1}{48\pi^2} [F_{L\mu\nu} \tilde{F}_L^{\mu\nu} + F_{R\mu\nu} \tilde{F}_R^{\mu\nu} + F_{L\mu\nu} \tilde{F}_R^{\mu\nu}]$$

Anomalies (cont'd):

[HEP-TH 0601155]

Covariant Forms:

Add a term to the lagrangian of the form $(1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$. The currents are now

modified to $\tilde{J} = J + \delta J$ and $\tilde{J}^5 = J^5 + \delta J^5$

$$\frac{\delta S'}{\delta V_\mu} = \delta J^\mu = -\frac{1}{3\pi^2}\epsilon_{\mu\nu\rho\sigma}A^\nu \partial^\rho V^\sigma + \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu \partial^\rho A^\sigma$$
$$\frac{\delta S'}{\delta A_\mu} = \delta J^{5\mu} = \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu \partial^\rho V^\sigma$$

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j \gg m^2$):

$$\partial^\mu \tilde{J}_\mu = 0$$
$$\partial^\mu \tilde{J}_\mu^5 = \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + \frac{1}{3}F_{A\mu\nu}\tilde{F}_A^{\mu\nu})$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j \ll m^2$):

$$\partial^\mu \tilde{J}_\mu = 0 \qquad \partial^\mu \tilde{J}_\mu = 0$$
$$\partial^\mu \tilde{J}_\mu^5 - 2im\bar{\psi}\gamma^5\psi = \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + \frac{1}{3}F_{A\mu\nu}\tilde{F}_A^{\mu\nu}) \qquad \partial^\mu \tilde{J}_\mu^5 = 0$$

Summary: Technically natural QED in D=5

Bulk: $D_A = \partial_A - iA_A, \quad F_{AB} = i[D_A, D_B], \quad L_0 = -\frac{1}{4\tilde{e}^2}F_{AB}F^{AB}$

$$\int_I d^4x \bar{\psi}_L i \not{D}_L \psi_L \quad \int_{II} d^4x \bar{\psi}_R i \not{D}_R \psi_R$$

Orbifold

$$D_{L\mu} = \partial_\mu - iA_\mu(x_\mu, 0) \quad D_{R\mu} = \partial_\mu - iA_\mu(x_\mu, R)$$

$$S_{CS} = \int d^5x \frac{1}{24\pi^2} \epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E$$

$$m\bar{\psi}_L(x_\mu, 0)W\psi_R(x_\mu, R) + h.c..$$

$$W = \exp(i \int_0^R A_5(x_\mu, x_5) dx^5)$$

Pass to $A_5 = 0$ Gauge

$$L_{CS} = \frac{3c}{4} \epsilon^{\mu\nu\rho\sigma} A_5 F_{\mu\nu} F_{\rho\sigma} + c \epsilon^{\mu\nu\rho\sigma} (\partial_5 A_\mu) A_\nu F_{\rho\sigma} .$$

Consider a Wilson line that emanates from, *e.g.*, brane I, $x^5 = 0$, toward an arbitrary point in the bulk, $x^5 = y$:

$$U(y) = \exp \left(i \int_0^y dx^5 A_5(x^5) \right) \quad \partial_y U = i A_5(y) U$$

Using the Wilson line as a gauge transformation, we have:

$$A_A \rightarrow A_A + iU^\dagger \partial_A U \quad A_5 \rightarrow A_5(y) + iU^\dagger \partial_y U = A_5(y) - \partial_y \int_0^y dx^5 A_5(x^5) = 0$$

$$B_\mu = A_\mu - \partial_\mu \int_0^y A_5 dx^5; \quad F_{B\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

$$S_{CS} = c \epsilon^{\mu\nu\rho\sigma} \int d^4 x \int_0^R dy (\partial_y B_\mu) B_\nu F_{B\rho\sigma}.$$

B_μ field will lead to gauge invariant

(“Stueckelberg”) combinations for each massive KK-mode in the compactified theory

The orbifold mode expansion

$$A_\mu^0(x, y) = \sqrt{\frac{1}{R}} \tilde{e} A_\mu^0(x)$$

$$A_\mu(x, y) = \sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{2}{R}} \tilde{e} \cos(n\pi y/R) A_\mu^n(x)$$

$$A_5(x, y) = \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt{\frac{2}{R}} \tilde{e} \sin(n\pi y/R) A_5^n(x)$$

$$S_1 = -\frac{1}{4\tilde{e}^2} \int_0^R dy \int d^4x F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \sum_n \int d^4x F_{\mu\nu}^n F^{n\mu\nu}$$

$$S_2 = \frac{1}{2\tilde{e}^2} \int_0^R dy \int d^4x F_{\mu 5} F^{\mu 5} = \frac{1}{2} \sum_{n=1} M_n^2 \int d^4x B_\mu^n B^{n\mu}$$

$$M_n = n\pi/R; \quad B_\mu^n = A_\mu^n + \frac{1}{M_n} \partial_\mu A_5^n; \quad F_{\mu\nu}^n \equiv \partial_\mu B_\nu^n - \partial_\nu B_\mu^n.$$

“Stueckelberg fields.”

$$e = \tilde{e}/\sqrt{R} \equiv e_0$$

$$e' = \sqrt{2}\tilde{e}/\sqrt{R} = \sqrt{2}e \equiv e_n \quad (n \neq 0)$$

$$\begin{aligned}
S_{CS} &= \frac{1}{24\pi^2} \int_0^R dy \int d^4x \epsilon^{\mu\nu\rho\sigma} (\partial_y B_\mu) B_\nu F_{\rho\sigma} \\
&\equiv \frac{1}{12\pi^2} \sum_{nmk} \int d^4x (e_n e_m e_k) c_{nmk} (B_\mu^n B_\nu^m \tilde{F}^{k\mu\nu}) \\
c_{nmk} &= (-1)^{(k+n+m)} \int_0^1 dz \partial_z [\cos(n\pi z)] \cos(m\pi z) \cos(k\pi z) \\
&= \frac{n^2(k^2 + m^2 - n^2) [(-1)^{(k+n+m)} - 1]}{(n+m+k)(n+m-k)(n-k-m)(n-m+k)}
\end{aligned}$$

$$c_{nm0} = c_{n0m} = -\frac{n^2}{n^2 - m^2} [(-1)^{n+m} - 1]$$

$$c_{0nm} = c_{00n} = 0$$

$$c_{n00} = [1 - (-1)^n].$$

D=4 Effective Theory

$$S_{full} = \int d^4x \left[\bar{\psi}(i\cancel{\partial} + V + \mathcal{A}\gamma^5 - m)\psi + \frac{1}{12\pi^2} \sum_{nmk} c_{nmk} B_\mu^n B_\nu^m \tilde{F}^{k\mu\nu} \right. \\ \left. - \frac{1}{4e^2} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{1}{4e'^2} \sum_{n \geq 1} F_{\mu\nu}^n F^{n\mu\nu} + \sum_{n \geq 1} \frac{1}{2e_n^2} M_n^2 B_\mu^n B^{n\mu} \right]$$

$$V_\mu = \sum_{n \text{ even}} B_\mu^n, \quad \mathcal{A}_\mu = \sum_{n \text{ odd}} B_\mu^n$$

if we truncate the theory on the zero mode B^0 and first KK-mode, B^1 ,

$$\frac{1}{12\pi^2} c_{100} B_\mu^1 B_\nu^0 \tilde{F}^{0\mu\nu} = \frac{1}{6\pi^2} \epsilon^{\mu\nu\rho\sigma} A_\mu V_\nu \partial_\rho V_\sigma$$

D=4 Effective Theory Current Algebra

$$\tilde{J}_\mu^n = \frac{\delta S}{\delta B^{n\mu}} = \bar{\psi}\gamma_\mu\psi|_{n \text{ even}} + \bar{\psi}\gamma_\mu\gamma^5\psi|_{n \text{ odd}} + J_\mu^{n \text{ CS}}$$

$$J_{mu}^{n \text{ CS}} = \frac{\epsilon_{\mu\nu\rho\sigma}}{12\pi^2} \sum_{mk} [(c_{nmk} - c_{mnk} + c_{kmn} - c_{mkn}) B^{m\nu} \partial^\rho B^{k\sigma}]$$

$$\partial^\mu J_\mu^n = \frac{1}{48\pi^2} \sum_{mk} (1 - (-1)^{n+m+k}) F_{\mu\nu}^m \tilde{F}^{k\mu\nu}$$

$$\partial^\mu J_\mu^{n \text{ CS}} = \frac{1}{48\pi^2} \sum_{m,k} (c_{nmk} - c_{mnk} + c_{nkm} - c_{knm}) F_{\mu\nu}^m \tilde{F}^{k\mu\nu}$$

KK-mode anomalies:

$$\partial^\mu \tilde{J}_\mu^n = \frac{1}{24\pi^2} \sum_{m,k} d_{nmk} F_{\mu\nu}^m \tilde{F}^{k\mu\nu}$$

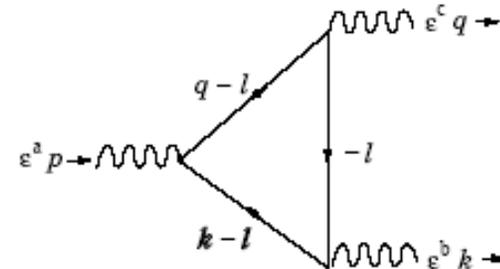
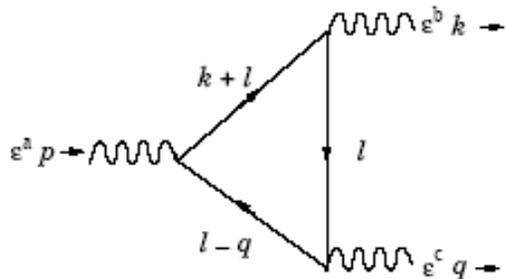
$$\begin{aligned} d_{nmk} &= \frac{1}{2} [(1 - (-1)^{n+m+k}) + (c_{nmk} - c_{mnk} + c_{nkm} - c_{knm})] \\ &= \frac{3}{2} [(-1)^{n+m+k} - 1] \frac{n^2(k^2 + m^2 - n^2)}{(k+m-n)(k+m+n)(k+n-m)(k-m-n)} \\ &= \frac{3}{2} c_{nmk} \quad \leftarrow \end{aligned}$$

$$c_{0mk} = 0 \quad \partial^\mu J_\mu^5 = \frac{1}{16\pi^2} \left(c_{100} F_{\gamma\mu\nu} \tilde{F}_\gamma^{k\mu\nu} + c_{111} F_{B\mu\nu} \tilde{F}_B^{k\mu\nu} \right) = \frac{1}{8\pi^2} F_{\gamma\mu\nu} \tilde{F}_\gamma^{\mu\nu} + \frac{1}{24\pi^2} F_{B\mu\nu} \tilde{F}_B^{\mu\nu}$$

D=4 Effective Theory in large m limit

$$S_{tree} = \int d^4x \left[\frac{1}{12\pi^2} \sum_{nmk} \bar{c}_{nmk} B_\mu^n B_\nu^m \tilde{F}^{k\mu\nu} - \frac{1}{4e^2} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{1}{4e'^2} \sum_{n \geq 1} F_{\mu\nu}^n F^{n\mu\nu} + \sum_{n=0} \frac{1}{2e_n^2} M_n^2 B_\mu^n B^{n\mu} \right]$$

Integrate out the Fermions:
Dirac Determinant effective interactions



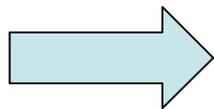
Dirac Determinant effective interaction:

$$\mathcal{O}_3 = -\frac{1}{12\pi^2} \epsilon^{\mu\nu\rho\sigma} \sum_{nmk} (e_n e_m e_k) a_{nmk} B_\mu^n B_\nu^m \partial_\rho B_\sigma^k$$

$$a_{nmk} = \frac{1}{2} (1 - (-1)^{n+m+k}) (-1)^{m+k}$$

This operator is equivalent to $(-1/6\pi^2) \epsilon_{\mu\nu\rho\sigma} A^\mu V^\nu \partial^\rho V^\sigma$

Dirac Determinant effective interaction
 equivalent To Bardeen's counterterm:
 consistent -> covariant

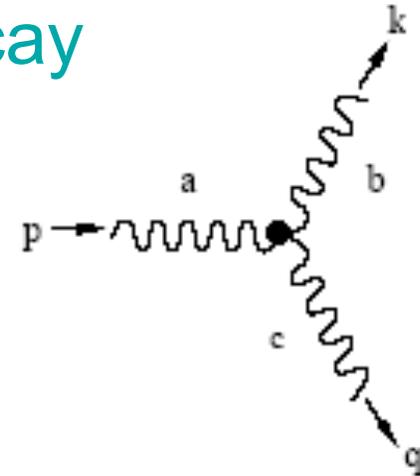


Full Effective Theory

$$\bar{c}_{nmk} = c_{nmk} - a_{nmk} \quad (\text{massive spinors})$$

$$\bar{c}_{nmk} = [(-1)^{(k+n+m)} - 1] \left(\frac{n^2(k^2 + m^2 - n^2)}{(n + m + k)(n + m - k)(n - k - m)(n - m + k)} + \frac{1}{2}(-1)^{m+k} \right)$$

Compute KK-Mode decay



Feynman rule for a vertex $B^a \rightarrow B^b + B^c$:

$$T_{CS} = -\frac{ee'^2}{12\pi^2} [(-\bar{c}_{abc} + \bar{c}_{bac} + \bar{c}_{bca} - \bar{c}_{cba})[B] + (\bar{c}_{acb} - \bar{c}_{cab} + \bar{c}_{bca} - \bar{c}_{cba})[A]]$$

$$[A] = \epsilon^{\mu\nu\rho\sigma} \epsilon_\mu^a \epsilon_\nu^b \epsilon_\rho^\gamma k^\sigma$$

$$[B] = \epsilon^{\mu\nu\rho\sigma} \epsilon_\mu^a \epsilon_\nu^b \epsilon_\rho^\gamma q^\sigma$$

A. Decay of KK-mode to KK-mode plus γ

$c = \text{photon}$

$$T_{CS} = -\frac{ee'^2}{12\pi^2} [(-\bar{c}_{ab0} + \bar{c}_{ba0} + \bar{c}_{b0a} - \bar{c}_{0ba})[B] + (\bar{c}_{a0b} - \bar{c}_{0ab} + \bar{c}_{b0a} - \bar{c}_{0ba})[A]]$$

$$= \frac{ee'^2}{2\pi^2} \left(\frac{M_b^2}{M_a^2 - M_b^2} - \frac{1}{2}((-1)^b - 1) \right) [B].$$

Gauge invariant in photon

$$\left. \begin{aligned} \Gamma_{1 \rightarrow 1+\gamma} &= \frac{2\alpha^3}{3\pi^3} \left(\frac{M_a^3}{M_b^2} \right) \\ \Gamma_{1+\rightarrow 1-\gamma} &= \frac{2\alpha^3}{3\pi^3} M_b \end{aligned} \right\} M_a^2 \gg M_b^2$$

$$\Gamma_{1\pm \rightarrow 1\mp \gamma} = \frac{2\alpha^3}{3\pi^3} \Delta M \quad \Delta M = M_a - M_b \ll M_a$$

$$\Gamma_{1 \rightarrow 1+\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3} \left(\frac{M_a}{R^2 M_b^2} \right) \quad \Gamma_{1+\rightarrow 1-\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3} \left(\frac{M_b}{R^2 M_a^2} \right)$$

B. Zero Mode + Zero Mode \rightarrow KK-Mode Vanishes

$$T_{CS} = -\frac{ee'^2}{12\pi^2} [(-\bar{c}_{a00} + \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[B] + (\bar{c}_{a00} - \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[A]]$$

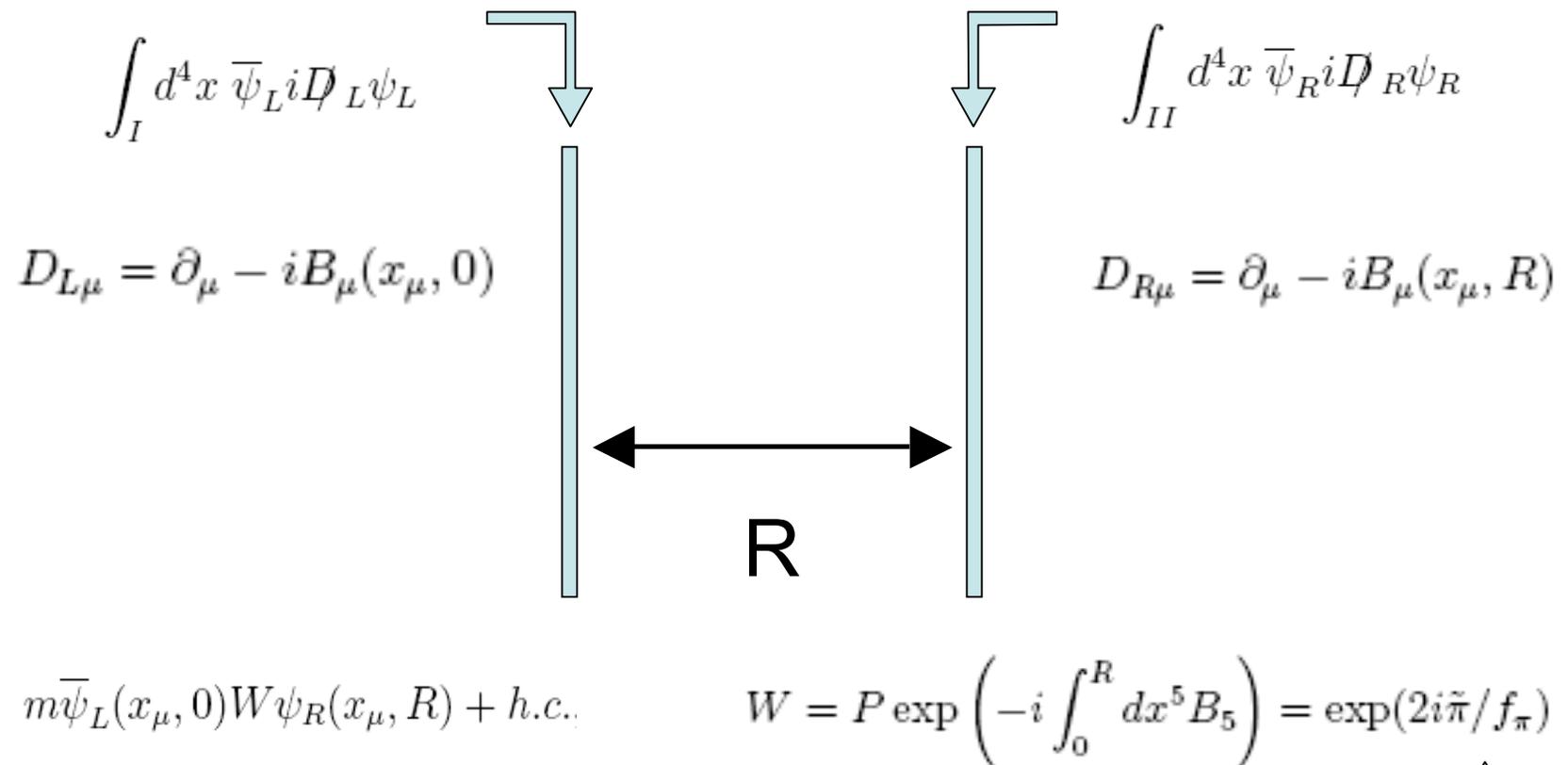
= 0 gauge invariance (Landau-Yang theorem)

Yang-Mills gauge theory of quark flavor in D=5:

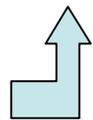
Bulk:

$$D_A = \partial_A - iB_A \quad B_A = B_A^a T^a,$$

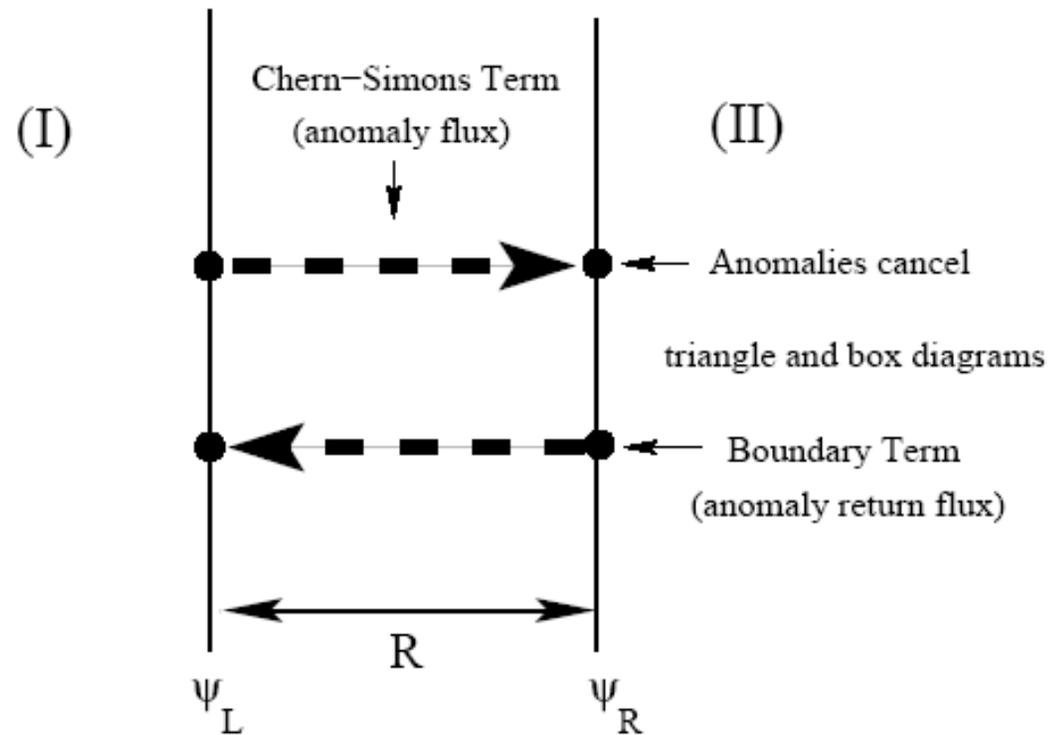
$$G_{AB} = i[D_A, D_B] = \partial_A B_B - \partial_B B_A - i[B_A, B_B].$$



Generic compactification; include B_5 zero-mode



Derivation of the full Wess-Zumino-Witten term directly from the Yang-Mills theory



Theory Requires Chern-Simons Term:

$$\begin{aligned}\mathcal{L}_{CS} &= c\epsilon^{ABCDE} \text{Tr}\left(A_A\partial_B A_C\partial_D A_E - \frac{3i}{2}A_A A_B A_C\partial_D A_E - \frac{3}{5}A_A A_B A_C A_D A_E\right) \\ &= \frac{c}{4}\epsilon^{ABCDE} \text{Tr}\left(A_A G_{BC} G_{DE} + iA_A A_B A_C G_{DE} - \frac{2}{5}A_A A_B A_C A_D A_E\right).\end{aligned}$$

Gauge transformation: $A_A \rightarrow V(A_A + i\partial_A)V^\dagger$ where: $V = \exp(i\theta^a T^a)$

$$\delta S_{CS} = c\epsilon^{\mu\nu\rho\sigma}\theta^a \text{Tr}\left[T^a(\partial_\mu A_\nu\partial_\rho A_\sigma - \frac{i}{2}(\partial_\mu A_\nu A_\rho A_\sigma - A_\mu\partial_\nu A_\rho A_\sigma + A_\mu A_\nu\partial_\rho A_\sigma))\right]\Big|_0^R$$

Consistent Anomaly; To cancel
against fermion anomalies:

$$c = \frac{N_c}{24\pi^2}.$$

Transforming to Axial Gauge, $B_5 \rightarrow 0$

$$V(x^\mu, y) = P \exp \left(-i \int_0^y dx^5 B_5^0(x^\mu, x^5) \right)$$

$$\psi'_L = \psi_L, \quad \psi'_R = V(R)\psi_R$$

$$\tilde{B}_\mu(x^\mu, y) = V(B_\mu + i\partial_\mu)V^\dagger \quad \tilde{B}_5(x^\mu, y) = V(B_5 + i\partial_y)V^\dagger = 0$$

$$\bar{\psi}(i\partial + \mathcal{B})\psi = \bar{\psi}'(i\partial + \tilde{\mathcal{B}})\psi' \quad \bar{\psi}_L W \psi_R = \bar{\psi}'_L \psi'_R \quad W \rightarrow V(0)WV^\dagger(R) = 1.$$

B is now a tower of vector mesons comingled
with the spin-0 mesons; must extract the physical mesons:

Redefinition: $\tilde{B}_\mu(x^\mu, y) = \tilde{U}(y)(A_\mu(x^\mu, y) + i\partial_\mu)\tilde{U}^\dagger(y) \quad \tilde{U} = \exp(2i\tilde{\pi}y/f_\pi)$

Compactification Decomposition of Chern-Simons Term:

$$\begin{aligned}
 S_{CS} &= c \int d^5x \epsilon^{ABCDE} \text{Tr} \left(B_A \partial_B B_C \partial_D B_E - \frac{3i}{2} B_A B_B B_C \partial_D B_E - \frac{3}{5} B_A B_B B_C B_D B_E \right) \\
 &= \frac{c}{2} \text{Tr} \int d^4x \int_0^R dy \left[(\partial_5 B_\mu) K^\mu + \frac{3}{2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} (B_5 G_{\mu\nu} G_{\rho\sigma}) \right],
 \end{aligned}$$

$$K^\mu \equiv \epsilon^{\mu\nu\rho\sigma} (i B_\nu B_\rho B_\sigma + G_{\nu\rho} B_\sigma + B_\nu G_{\rho\sigma}).$$



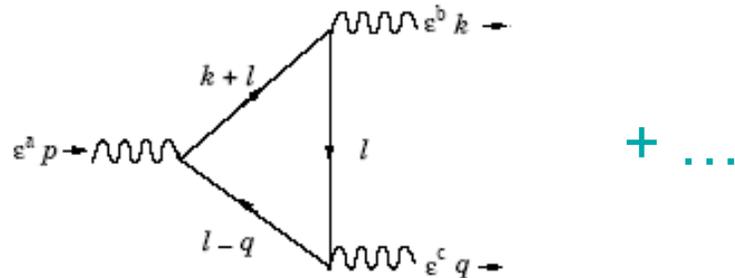
Axial Gauge:

CTH and Zachos

$$\begin{aligned}
 S_{CS} &= \frac{c}{2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \int_0^R dy \text{Tr} \left[\partial_y \tilde{B}_\mu \left(i \tilde{B}_\nu \tilde{B}_\rho \tilde{B}_\sigma + G(\tilde{B})_{\nu\rho} \tilde{B}_\sigma + \tilde{B}_\nu G(\tilde{B})_{\rho\sigma} \right) \right] \\
 &= \frac{c}{2} \text{Tr} \int d^4x \int_0^R dy (\partial_y \tilde{B}) (2d\tilde{B}\tilde{B} + 2\tilde{B}d\tilde{B} - 3i\tilde{B}^3)
 \end{aligned}$$

Form notation: $G(\tilde{B}) = 2d\tilde{B} - 2i\tilde{B}^2.$

Integrate out the Fermions: Dirac Determinant effective interactions



Dirac Determinant effective interaction
Is the Bardeen Counterterm as in QED:

$$S_{\text{boundary}} = -\frac{c}{2} \int \text{Tr} \left(\frac{1}{2} (G_R \tilde{B}_R + \tilde{B}_R G_R) \tilde{B}_L - \frac{1}{2} (G_L \tilde{B}_L + \tilde{B}_L G_L) \tilde{B}_R \right. \\ \left. + i \tilde{B}_R^3 \tilde{B}_L - i \tilde{B}_L^3 \tilde{B}_R - \frac{i}{2} (\tilde{B}_R \tilde{B}_L)^2 \right)$$

“Boundary term”

Notation:

$$\tilde{B}_\mu = \tilde{A}_\mu - i\alpha_\mu \quad \tilde{A}_\mu = \tilde{U} A_\mu \tilde{U}^\dagger$$

$$\tilde{A}_{L\mu} = A_{L\mu} = A_\mu(x^\mu, 0) \quad \tilde{A}_{R\mu} = U A_\mu(x^\mu, R) U^\dagger$$

$$\alpha_\mu = -\tilde{U} \partial_\mu \tilde{U}^\dagger \quad \beta_\mu = U^\dagger \partial_\mu U = U^\dagger \alpha_\mu U \quad \tilde{U} = \exp(2i\tilde{\pi}y/f_\pi)$$

$$\begin{aligned} S_{CS} \Rightarrow & \frac{c}{2} \text{Tr} \int d^4x dy [-i(\partial_y \alpha) + (\partial_y \tilde{A})] \\ & \times (2d\tilde{A}\tilde{A} - 2i\alpha^2 \tilde{A} - 2id\tilde{A}\alpha - 4\alpha^3 + 2\tilde{A}d\tilde{A} - 2i\tilde{A}\alpha^2 - 2i\alpha d\tilde{A} \\ & - 3i\tilde{A}^3 - 3\alpha\tilde{A}^2 - 3\tilde{A}\alpha\tilde{A} - 3\tilde{A}^2\alpha + 3i\alpha^2 \tilde{A} + 3i\alpha\tilde{A}\alpha + 3i\tilde{A}\alpha^2 + 3\alpha^3) \end{aligned}$$

$$\begin{aligned} S_{\text{boundary}} \Rightarrow & \frac{c}{2} \int \text{Tr} [(dA_L A_L + A_L dA_L) U A_R U^\dagger - (dA_R A_R + A_R dA_R) U^\dagger A_L U \\ & - i(dA_L A_L + A_L dA_L) \alpha - A_L^3 \alpha - A_L \alpha^3 + iA_R^3 U^\dagger A_L U - iA_L^3 U A_R U^\dagger \\ & - i(dA_R dU^\dagger A_L U - dA_L dU A_R U^\dagger) - (A_R U^\dagger A_L U A_R \beta + A_L U A_R U^\dagger A_L \alpha) \\ & + \frac{i}{2} A_L \alpha A_L \alpha + \frac{i}{2} U A_R U^\dagger A_L U A_R U^\dagger A_L - i(A_L U A_R U^\dagger \alpha^2 - A_R U^\dagger A_L U \beta^2)] \end{aligned}$$

We first isolate the term:

$$S_{CS0} = i\frac{c}{2} \text{Tr} \int (\partial_y \alpha) \alpha^3$$

$$\partial_y \alpha = \partial_y \tilde{U} d\tilde{U}^\dagger = \frac{2i}{Rf_\pi} \tilde{U} d\tilde{\pi} \tilde{U}^\dagger \quad \alpha \approx \frac{2iy}{f_\pi} d\tilde{\pi} - \frac{2y^2}{f_\pi^2} [\tilde{\pi}, d\tilde{\pi}] + \dots$$

$$\begin{aligned} S_{CS0} &= -\frac{2N_c}{3\pi^2 f_\pi^5} \int d^4x dy y^4 \text{Tr}(\tilde{\pi} d\tilde{\pi} d\tilde{\pi} d\tilde{\pi} d\tilde{\pi}) + \dots \\ &= -\frac{2N_c}{15\pi^2 f_\pi^5} \int d^4x \text{Tr}(\tilde{\pi} d\tilde{\pi} d\tilde{\pi} d\tilde{\pi} d\tilde{\pi}) + \dots \end{aligned}$$

$$\begin{aligned}
S_{CS \alpha^3 \tilde{A}} &= -i \frac{c}{2} \text{Tr} \int (\partial_y \alpha) (-2i d\tilde{A}\alpha - 2i\alpha d\tilde{A} - 2i\alpha^2 \tilde{A} - 2i\tilde{A}\alpha^2 + 3i(\alpha^2 \tilde{A} + \alpha \tilde{A}\alpha + \tilde{A}\alpha^2)) \\
&\quad - \frac{c}{2} \text{Tr} \int (\partial_y \tilde{A}) [\alpha^3]
\end{aligned} \tag{42}$$

Note that, upon integrating in $D = 4$ by parts:

$$\text{Tr} \int (\partial_y \alpha) (d\tilde{A}\alpha + \alpha d\tilde{A}) = 2 \text{Tr} \int (\partial_y \alpha) (\alpha \tilde{A} \alpha)$$

Thus, we can immediately write:

$$\begin{aligned}
S_{CS \alpha^3 \tilde{A}} &= -i \frac{c}{2} \text{Tr} \int (\partial_y \alpha) (i\alpha^2 \tilde{A} + i\tilde{A}\alpha^2 - i\alpha \tilde{A}\alpha) - \frac{c}{2} \text{Tr} \int d^4 x dy (\partial_y \tilde{A}) [\alpha^3] \\
&= \frac{c}{2} \text{Tr} \int d^4 x \int_0^1 dy \partial_y (\alpha^3 \tilde{A})
\end{aligned}$$

If we now explicitly perform this integral we obtain:

$$S_{CS \alpha^3 \tilde{A}} = -\frac{c}{2} \text{Tr}(A_R \beta^3)$$

where use has been made $\text{Tr}(\alpha^3 \tilde{A}_R) = \text{Tr}(\alpha^3 U A_R U^\dagger) = \text{Tr}(U^\dagger \alpha^3 U A_R) = \text{Tr}(\beta^3 A_R) = -\text{Tr}(A_R \beta^3)$. We see the operational parity asymmetry of our gauge transformation leads to the absence of a corresponding parity conjugate term, $-\text{Tr}(A_L \alpha^3)$. As mentioned above, this term will come from the boundary term, and the overall final result will be parity symmetric.

Obtain the full Wess-Zumino-Witten Term

$$\tilde{S} = S_{CS} + S_{boundary} = S_{WZW} + S_{bulk}$$

$$\begin{aligned} S_{WZW} = S_{CS0} &+ \frac{N_c}{48\pi^2} \text{Tr} \int d^4x [-(A_L\alpha^3 + A_R\beta^3) - (A_L^3\alpha + A_R^3\beta) \\ &- i((dA_L A_L + A_L dA_L)\alpha + dA_R A_R + A_R dA_R)\beta) + \frac{i}{2}[(A_L\alpha)^2 - (A_R\beta)^2] \\ &- i(A_L^3 U A_R U^\dagger - A_R^3 U^\dagger A_L U) \\ &+ (dA_L A_L + A_L dA_L)U A_R U^\dagger - (dA_R A_R + A_R dA_R)U^\dagger A_L U \\ &- i(dA_R dU^\dagger A_L U - dA_L dU A_R U^\dagger) - (A_L U A_R U^\dagger A_L \alpha + A_R U^\dagger A_L U A_R \beta) \\ &+ \frac{i}{2}U A_R U^\dagger A_L U A_R U^\dagger A_L - i(A_L U A_R U^\dagger \alpha^2 - A_R U^\dagger A_L U \beta^2)] \end{aligned}$$

$$\tilde{S}_{CS0} = -\frac{2N_c}{15\pi^2 f_\pi^5} \int d^4x \text{Tr}(\tilde{\pi} d\tilde{\pi} d\tilde{\pi} d\tilde{\pi} d\tilde{\pi}) + \dots$$

in complete agreement with Kaymakçalan, Rajeev and Schechter

Effective brane (holographic) interaction

Normal Derivation of WZW term:

- promote full theory of mesons to $D=5$.
- In $D=5$, a certain manifestly chirally invariant and topologically interesting Chern-Simons term occurs, which is included into the theory.
- Compactify the fifth dimension with the Chern-Simons term, back into to $D=4$, resulting in the Wess-Zumino term.
- Perform gauge transformations upon the resulting object, and infer how to ``integrate in'' the gauge fields by brute force and some guess work.

$$S_{bulk} = -i\frac{c}{2} \text{Tr} \int (\partial_y \alpha) (\tilde{U} (3dAA + 3AdA - 4iA^3) \tilde{U}^\dagger) \\ + \frac{c}{2} \text{Tr} \int (\partial_y \tilde{A}) [\tilde{U} (2dAA + 2AdA - 3iA^3) \tilde{U}^\dagger]$$

$$\partial_y \alpha = \frac{2i}{f_\pi} \tilde{U} (d\tilde{\pi}) \tilde{U}^\dagger \quad \partial_y \tilde{A} = \partial_y \tilde{U} A \tilde{U}^\dagger = \frac{2i}{f_\pi} \tilde{U} ([\tilde{\pi}, A]) \tilde{U}^\dagger$$

$$S_{bulk} = -\frac{3c}{2f_\pi} \int d^4x \int_0^1 dy \text{Tr}(\tilde{\pi}GG) + \frac{c}{2} \int d^4x \int_0^1 dy \text{Tr}(\partial_y A)(2dAA + 2AdA - 3iA^3)) \quad (6)$$

Effective bulk interaction

Suppose we don't integrate out the quarks?

Parity symmetric redefinition field: $\tilde{U}(y) = \exp\left(\frac{2i\tilde{\pi}(y - 1/2)}{f_\pi}\right)$

$$\tilde{B}_L = \xi A_L \xi^\dagger - j_L \quad \tilde{B}_R = \xi^\dagger A_L \xi - j_R$$

chiral currents $j_L = i\xi d\xi^\dagger \quad j_R = -i\xi^\dagger d\xi$

$$\begin{aligned}
 S &= S_{CS0} + S'_{WZW} + S_{bulk} \\
 &+ \int_I d^4x \bar{\psi}_L (i\cancel{\partial} + \xi A_L \xi^\dagger - j_L) \psi_L + \int_{II} d^4x \bar{\psi}_R (i\cancel{\partial} + \xi^\dagger A_R \xi - j_R) \psi_R \\
 S'_{WZW} &= -\frac{c}{2} \text{Tr}(A_R j_R^3 + A_L j_L^3) - \frac{c}{2} \text{Tr}(A_R^3 j_R + A_L^3 j_L) - i\frac{c}{4} \text{Tr}(A_R j_R A_R j_R - A_L j_L A_L j_L) \\
 &\quad - i\frac{c}{2} \text{Tr}[(dA_R A_R + A_R dA_R) j_R + (dA_L A_L + A_L dA_L) j_L]
 \end{aligned} \tag{67}$$

Effective theory with unintegrated massless fermions

Envisioned applications:

- (1) Little Higgs Theories.
- (2) RS Models
- (3) A WZW Term for the Goldstone-Wilczek Current
- (4) Skyrme/instanton baryogenesis/b+L violation
in extra dimensional theories
(which is what started this project).
- (5) Technical tool to analyze/generalize WZW term structure
- (6) Can apply to $D=3$ YM matching to $D=2$;
- (7) Any $D+1$ to D .
- (8) ...