Fermilab

Theory Seminar

February 2007

# Constructing One-Loop Amplitudes

Ruth Britto

University of Amsterdam

## Subject

One-loop amplitudes in QCD, as the virtual contribution and “bottleneck” to computing hard-scattering cross sections at NLO.

Fully analytic approach:

- possibly competitive with partly numerical methods
- explore more formal features of perturbation theory

## Status

$2 \longrightarrow 3$ processes are state of the art in QCD at NLO.

Complexity increases with number of kinematic variables. Last year, complexity of $2 \longrightarrow 4$ was tackled with results for the 6-gluon amplitude:

- Semi-numerical method, Feb. 2006

  Ellis, Giele, Zanderighi

- Analytic results completed, July 2006

  Bern, Dixon, Dunbar, Kosower (x2); Bidder, Bjerrum-Bohr, Dixon, Dunbar; RB, Buchbinder, Cachazo, Feng; Bidder, Bjerrum-Bohr, Dunbar, Perkins (x2); Bedford, Brandhuber, Spence, Travaglini; Bern, Bjerrum-Bohr, Dunbar, Ita; Bern, Dixon, Kosower; RB, Feng, Mastrolia; Berger, Bern, Dixon, Forde, Kosower (x2); Xiao, Yang, Zhu

## Cut construction

One-loop amplitudes in gauge theory have long been known to be "cut-constructible", i.e. fully determined by unitarity cuts, but concrete methods of carrying out the construction have become available only recently.

In this talk I present a systematic procedure to construct one-loop amplitudes in gauge theory from their unitarity cuts.

There are no restrictions on helicities.

Our principal new tool is spinor integration. This procedure is in fact algebraic.$^\star$

Although spinors are naturally suited to massless particles in four dimensions, the formalism applies to more general amplitudes.

## no twistors here

twistor string theory

$\downarrow$

twistor geometry

$\downarrow$

good old spinors

# Contents

- Preliminaries: color decomposition & basis of integrals
- Unitarity cuts
- Quadruple cuts for box integrals
- Spinor-helicity formalism
- Spinor integration for ordinary unitarity cuts
- Dimensional regularization

## Color Decomposition

(Berends, Giele; Mangano, Parke, Xu; Mangano; Bern, Kosower)

$$A_n^{1-\mathrm{loop}}(\{p_i^\mu, \epsilon_i^\mu, a_i\}) =$$

$$g^N \sum_{\sigma \in S_n/Z_n} N_c \, \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A_{n;1}(\sigma(p_1^\mu, \epsilon_1^\mu), \ldots, \sigma(p_n^\mu, \epsilon_n^\mu))$$

$$+ g^N \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}}\right) \mathrm{Tr}\left(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}}\right)$$

$$\times A_{n;c}(\sigma(p_1^\mu, \epsilon_1^\mu), \ldots, \sigma(p_n^\mu, \epsilon_n^\mu))$$

**“color ordered partial amplitudes”**

Need only leading-color partial amplitudes $A_{n;1}$; these determine the others. (Bern, Dixon, Dunbar, Kosower)

Retain only cyclic ordering.

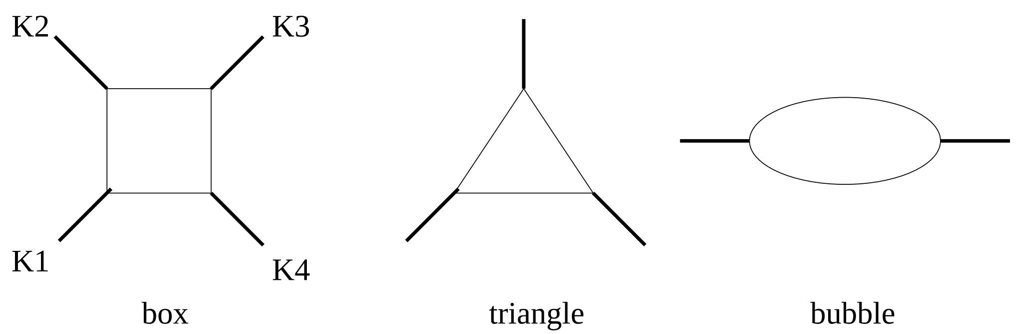
Convention: All momenta are incoming.

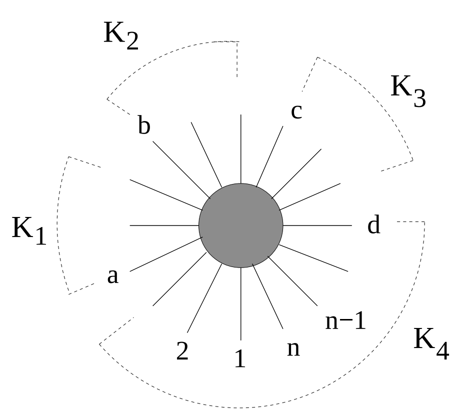
## Basis of Master Integrals

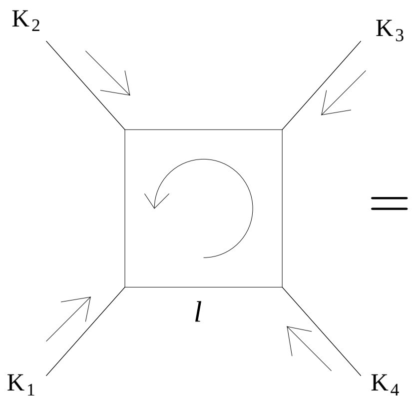
Passarino-Veltman reduction brings the one-loop amplitude to the form

$$A_{n;1} = \sum_i d_i \text{ (box)} + \sum_i c_i \text{ (triangle)} + \sum_i b_i \text{ (bubble)} + \text{rational}$$

where expressions for scalar bubble, scalar triangle and scalar box integrals are known explicitly. (in dim. reg.: Bern, Dixon, Kosower )





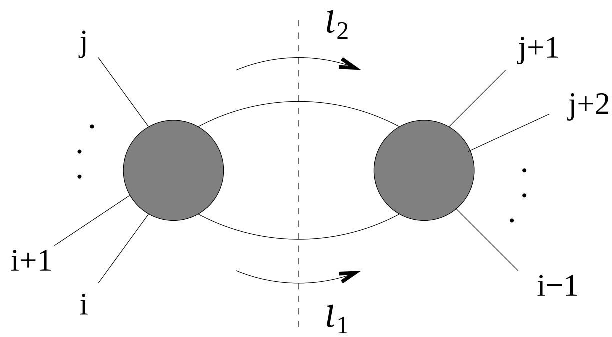

$$= \int d^{4-2\epsilon}\ell \frac{1}{\ell^2(\ell - K_1)^2(\ell - K_1 - K_2)^2(\ell + K_4)^2}$$

- Depending on how many massive legs, we have $I_4^{4m}$, $I_4^{3m}$, $I_4^{2m\ e}$, $I_4^{2m\ h}$, $I_4^{1m}$, $I_3^{3m}$, $I_3^{2m}$, $I_3^{1m}$ and $I_2^{2m}$.

  e.g. $I_3^{2m}$ has 3 legs ("triangle integral") of which 2 are massive.

- So the problem is reduced to finding the coefficients $b_i, c_i, d_i$, which are rational functions of the spinor products $\langle i\ j\rangle$ and $[i\ j]$ (plus the rational piece, which we will deal with later).

# Unitarity Cuts




$$C = \int d\mu \; A_L^{\text{tree}} \, A_R^{\text{tree}}$$

The measure is

$$d\mu = d^4\ell_1 \, d^4\ell_2 \, \delta^{(4)}(\ell_1 + \ell_2 - K) \, \delta^{+}(\ell_1^2) \, \delta^{+}(\ell_2^2).$$

Drop the principal part of the two Feynman propagators, leaving the delta function that places their momenta on-shell.

This quantity is the discontinuity $\Delta$ of the amplitude across the corresponding branch cut.

$$C = \Delta A_n^{1-\mathrm{loop}}$$

By unitarity, this is the imaginary part of the amplitude in the kinematic regime where $K^2 > 0$ and all other momentum invariants are negative.

Unitarity methods use “cut constructibility” in terms of the master integrals.

(Bern, Dixon, Dunbar, Kosower; Cachazo)

$$C = \Delta A_n^{1-\mathrm{loop}} = \sum c\, \Delta I$$

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We still get several coefficients together in the same equation.

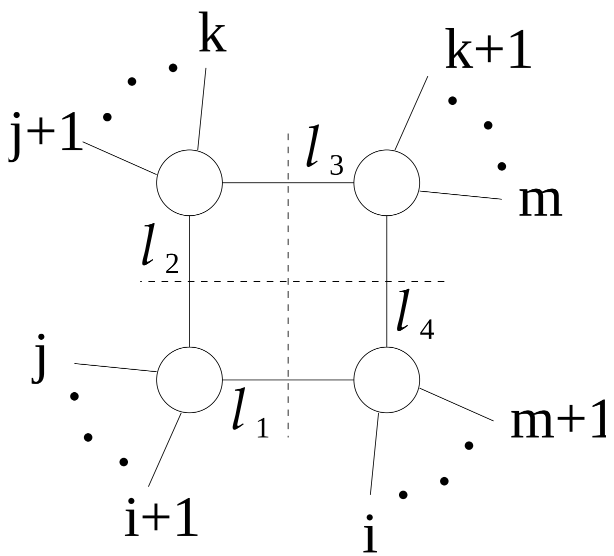
Earlier versions of the “unitarity method” involved integral reduction and an intelligent ansatz for a given coefficient based on singularities. (Bern, Dixon, Dunbar, Kosower; Bern, Del Duca, Dixon, Kosower)

Recently we learned how to extract any given coefficient directly, systematically and algebraically.

Box coefficients can be isolated individually using quadruple cuts, which are particularly simple to evaluate in four dimensions.

# Box Coefficients from Quadruple Cuts

(RB, Cachazo, Feng)




There is a notion of generalized unitarity. The box diagram has a leading singularity whose discontinuity is given by replacing all four propagators by their on-shell delta functions. This leading singularity picks out a given box uniquely.

We are in four dimensions, so four delta functions localize the integral completely. This computation is very easy!

## The solutions of loop momenta

The box coefficients computed from quadruple cuts are given by

$$\frac{1}{2}\sum_{\mathcal{S}} A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$

$\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines.

$$\mathcal{S} = \{\, \ell \,|\, \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0\}$$

Can these equations always be solved?

In complexified momentum space, there are exactly 2 solutions.

(Moreover, 3-point amplitudes do not always vanish.)

We have learned to distinguish the contributions from boxes, triangles and bubbles even when they participate in the same unitarity cut. The cut boxes and triangles have logarithms of unique “signature” functions of kinematic invariants, while the cut bubbles are entirely rational.

So there is no strict need for generalized cuts of one-loop amplitudes.

It is particularly efficient to get box coefficients from quadruple cuts. However, it is not obvious that it saves work to separate triangle coefficients through triple cuts.

Let us simply understand how to carry out the integration of double cuts.

Techniques have improved by revisiting the spinor-helicity formalism.

## Spinor-Helicity Formalism

Berends, Kleiss, De Causmaecker, Gastmans, Wu (1981); De Causmaecker, Gastmans, Troost, Wu (1982); Kleiss, Stirling (1985); Gastmans, Wu (1990); Xu, Zhang, Chang (1987); Gunion, Kunszt (1985)

Lorentz group $\sim SL(2) \times SL(2)$, so we have spinors of positive and negative chirality, $\lambda_a$ of $(\frac{1}{2}, 0)$ and $\tilde{\lambda}_{\dot{a}}$ of $(0, \frac{1}{2})$ with $a, \dot{a} = 1, 2$.

General 4-vector: $P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \lambda'_a \tilde{\lambda}'_{\dot{a}}$.

Null vector: $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$.

Lorentz invariant spinor products:

$$\langle 1\,2\rangle \quad \equiv \quad \langle \lambda_1\,\lambda_2\rangle \equiv \epsilon_{ab}\lambda_1^a\lambda_2^b$$

$$[1\,2] \quad \equiv \quad [\tilde\lambda_1\,\tilde\lambda_2] \equiv \epsilon_{\dot a\dot b}\tilde\lambda_1^{\dot a}\tilde\lambda_2^{\dot b}$$

$$(\langle 1\,2\rangle\,[1\,2] = s_{12})$$

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$$\langle 1\,2\rangle \quad\equiv\quad \langle \lambda_1\;\lambda_2\rangle \equiv \epsilon_{ab}\lambda_1^a\lambda_2^b$$
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$$(\langle 1\,2\rangle\,[1\,2] = s_{12})$$

Helicities of gluons in terms of spinors:

$$- : \;\epsilon_{a\dot a} = \frac{\lambda_a\tilde\mu_{\dot a}}{[\tilde\lambda\;\tilde\mu]} \qquad + : \;\tilde\epsilon_{a\dot a} = \frac{\tilde\lambda_{\dot a}\mu_a}{\langle\mu\;\lambda\rangle}$$

where $\mu$ and $\tilde\mu$ are arbitrary.

Lorentz invariant spinor products:

$$\begin{aligned} \langle 1\ 2\rangle &\equiv \langle \lambda_1\ \lambda_2\rangle \equiv \epsilon_{ab}\lambda_1^a\lambda_2^b \\ [1\ 2] &\equiv [\tilde\lambda_1\ \tilde\lambda_2] \equiv \epsilon_{\dot a\dot b}\tilde\lambda_1^{\dot a}\tilde\lambda_2^{\dot b} \end{aligned}$$

$$(\langle 1\ 2\rangle\,[1\ 2] = s_{12})$$

Helicities of gluons in terms of spinors:

$$- :\ \epsilon_{a\dot a} = \frac{\lambda_a\tilde\mu_{\dot a}}{[\tilde\lambda\ \tilde\mu]} \qquad + :\ \tilde\epsilon_{a\dot a} = \frac{\tilde\lambda_{\dot a}\mu_a}{\langle\mu\ \lambda\rangle}$$

where $\mu$ and $\tilde\mu$ are arbitrary.

More notation:

$$\begin{aligned} P_{i,j} &\equiv p_i + p_{i+1} + \cdots + p_j \\ \langle a|P_{i,j}|b] &\equiv \sum_{r=i}^{j}\langle a\ r\rangle\,[r\ b] \end{aligned}$$

## MHV tree amplitudes

(Parke, Taylor; Berends, Giele)

$$A(1^+, 2^+, \ldots, j^-, \ldots, k^-, \ldots, n-1^+, n^+)$$
$$= \frac{\langle j\, k\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n-1\; n\rangle\langle n\, 1\rangle}$$

## Complexified Momenta – Spacetime Signatures

In real Minkowski space, the spinors $\lambda_a$ and $\widetilde{\lambda}_{\dot{a}}$ are complex-valued and conjugate to each other.

It is useful to consider the spinors $\lambda_a$ and $\widetilde{\lambda}_{\dot{a}}$ as complex-valued and independent. One can think of this as complexified momentum space.

This means that we treat the amplitude as a complex function of the spinor products $\langle\lambda\,\lambda'\rangle$ and $[\widetilde{\lambda}\,\widetilde{\lambda}']$. We can examine its analytic structure. At the end, we may specialize to physical momentum space.

This approach allowed us to propose and derive on-shell recursion relations for tree amplitudes.

# Recursion Relations for Tree Amplitudes of Gluons

(RB, Cachazo, Feng; RB, Cachazo, Feng, Witten)

Derived from residue theorem after introducing a complex variable $z$.

Amplitude expressed in terms of factorization limits.

$$A_n(1,\ldots,n{-}2,(n{-}1)^-,n^+) =$$

$$\sum_{i=1}^{n-3}\sum_{h=+,-} A_{i+2}(\hat{n},1,\ldots,i,-\hat{P}^h_{n,i})\frac{1}{P^2_{n,i}}A_{n-i}(+\hat{P}^{-h}_{n,i},i{+}1,\ldots,n{-}2,\widehat{n-1})$$

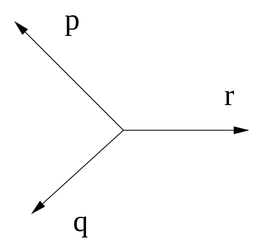
where

$$P_{n,i} = p_n + p_1 + \ldots + p_i, \quad \hat{P}_{n,i} = P_{n,i} + \frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}\tilde{\lambda}_n,$$

$$\hat{\tilde{\lambda}}_{n-1} = \tilde{\lambda}_{n-1} - \frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\tilde{\lambda}_n, \quad \hat{\lambda}_n = \lambda_n + \frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}.$$

## Complexified Momenta and Three-Gluon Amplitudes

(Witten)




$$r^2 = 0 \Rightarrow p \cdot q = 0.$$

$$2p \cdot q = \langle \lambda_p \, \lambda_q \rangle [\tilde{\lambda}_p \, \tilde{\lambda}_q] \Rightarrow \langle \lambda_p \, \lambda_q \rangle = 0 \quad \text{or} \quad [\tilde{\lambda}_p \, \tilde{\lambda}_q] = 0.$$

That is, $\lambda_p \sim \lambda_q$ or $\tilde{\lambda}_p \sim \tilde{\lambda}_q$.

In Minkowski signature with real momenta, $\tilde{\lambda} = \pm \bar{\lambda}$, so we have
both $\langle \lambda_p \, \lambda_q \rangle = 0$ and $[\tilde{\lambda}_p \, \tilde{\lambda}_q] = 0$.

$$A(p^+,q^+,r^-) = \frac{[\tilde\lambda_p\,\tilde\lambda_q]^3}{[\tilde\lambda_q\,\tilde\lambda_r][\tilde\lambda_r\,\tilde\lambda_p]} \qquad A(p^-,q^-,r^+) = \frac{\langle\lambda_p\,\lambda_q\rangle^3}{\langle\lambda_q\,\lambda_r\rangle\langle\lambda_r\,\lambda_p\rangle}$$

In Minkowski signature with real momenta, both of these amplitudes vanish.

Working with complexified momenta, these vertices do not always vanish.

They just select one of the two conditions of proportional spinors.

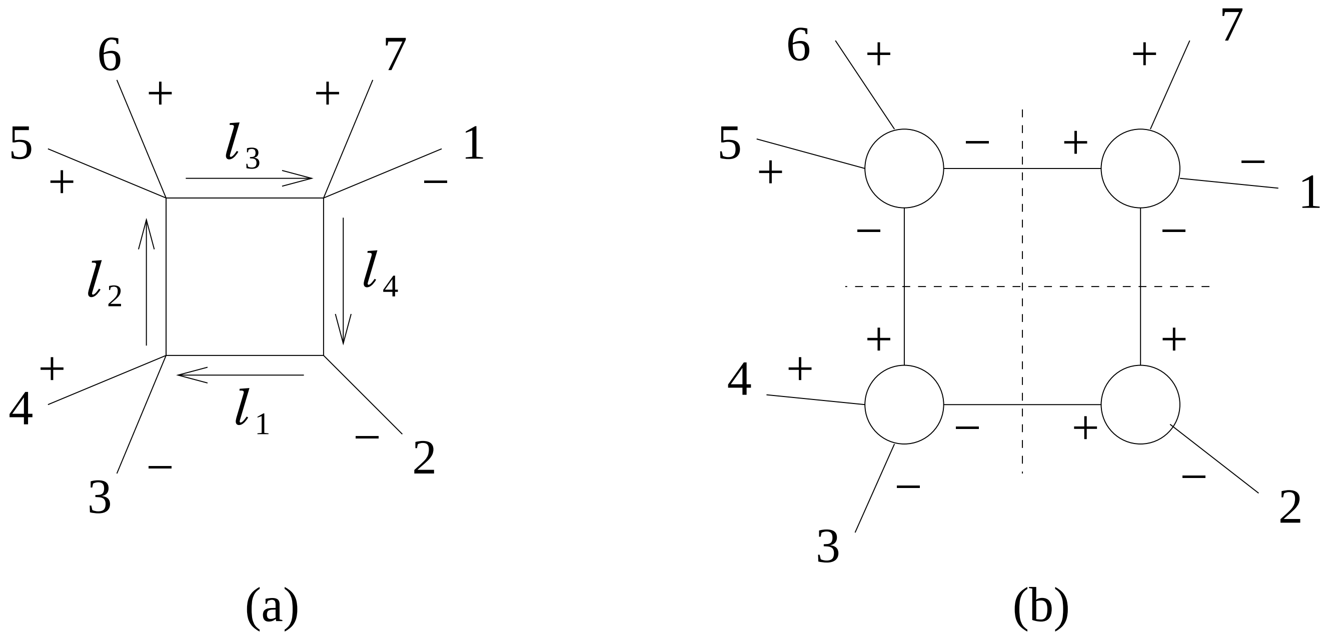
$$A(p^+,q^+,r^-): \qquad \langle\lambda_p\,\lambda_q\rangle = \langle\lambda_p\,\lambda_r\rangle = \langle\lambda_q\,\lambda_r\rangle = 0$$

$$(\lambda_p \sim \lambda_q \sim \lambda_r)$$

$$A(p^-,q^-,r^+): \qquad [\tilde\lambda_p\,\tilde\lambda_q] = [\tilde\lambda_p\,\tilde\lambda_r] = [\tilde\lambda_q\,\tilde\lambda_r] = 0$$

$$(\tilde\lambda_p \sim \tilde\lambda_q \sim \tilde\lambda_r)$$

# Box Coefficient from Quadruple Cut




$$\text{coeff} = \frac{1}{2}\,\frac{[\ell_1\,\ell_4]^3}{[\ell_1\,2][2\,\ell_4]}\,\frac{[4\,\ell_2]^3}{[\ell_2\,\ell_1][\ell_1\,3][3\,4]}\,\frac{[5\,6]^3}{[6\,\ell_3][\ell_3\,\ell_2][\ell_2\,5]}\,\frac{[\ell_3\,7]^3}{[7\,1][1\,\ell_4][\ell_4\,\ell_3]}$$

$$\text{coeff} = \frac{1}{2} \frac{[\ell_1\ \ell_4]^3}{[\ell_1\ 2][2\ \ell_4]} \frac{[4\ \ell_2]^3}{[\ell_2\ \ell_1][\ell_1\ 3][3\ 4]} \frac{[5\ 6]^3}{[6\ \ell_3][\ell_3\ \ell_2][\ell_2\ 5]} \frac{[\ell_3\ 7]^3}{[7\ 1][1\ \ell_4][\ell_4\ \ell_3]}$$

Multiply and divide by $\langle \ell_4\ \ell_1\rangle^3 \langle \ell_2\ \ell_1\rangle^3 \langle \ell_1\ \ell_3\rangle^3$.

$$\frac{1}{2} \frac{[\ell_1|\ell_4|\ell_1\rangle^3}{[\ell_1\ 2]\ [2|\ell_4|\ell_1\rangle} \frac{[4|\ell_2|\ell_1\rangle^3}{\langle \ell_1|\ell_2|\ell_1]\ [\ell_1\ 3][3\ 4]} \frac{[5\ 6]^3}{[6|\ell_3|\ell_1\rangle\ \langle \ell_1|\ell_3|\ell_2|\ell_1\rangle\ \langle \ell_1|\ell_2|5]} \frac{\langle \ell_1|\ell_3|7]^3}{[7\ 1]\ [1|\ell_4|\ell_1\rangle\ \langle \ell_1|\ell_4|\ell_3|\ell_1\rangle}$$

Now $\ell_2, \ell_3, \ell_4$ appear as vectors and can be expressed in terms of $\ell_1$.

$$\mathrm{coeff} = \frac{1}{2} \frac{[\ell_1\ \ell_4]^3}{[\ell_1\ 2][2\ \ell_4]} \frac{[4\ \ell_2]^3}{[\ell_2\ \ell_1][\ell_1\ 3][3\ 4]} \frac{[5\ 6]^3}{[6\ \ell_3][\ell_3\ \ell_2][\ell_2\ 5]} \frac{[\ell_3\ 7]^3}{[7\ 1][1\ \ell_4][\ell_4\ \ell_3]}$$

Multiply and divide by $\langle\ell_4\ \ell_1\rangle^3\langle\ell_2\ \ell_1\rangle^3\langle\ell_1\ \ell_3\rangle^3$.

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Now $\ell_2, \ell_3, \ell_4$ appear as vectors and can be expressed in terms of $\ell_1$.

Expand $|\ell_1\rangle = \alpha|2\rangle + \beta|3\rangle, \ \ |\ell_1] = \gamma|2] + \delta|3]$.

Solve $0 = \ell_2^2 = \ell_3^2 = \ell_4^2$ for $\alpha, \beta, \gamma, \delta$. (Remaining freedom is guaranteed to drop out.)

$$\text{coeff} = \frac{1}{2}\frac{[\ell_1\ \ell_4]^3}{[\ell_1\ 2][2\ \ell_4]}\frac{[4\ \ell_2]^3}{[\ell_2\ \ell_1][\ell_1\ 3][3\ 4]}\frac{[5\ 6]^3}{[6\ \ell_3][\ell_3\ \ell_2][\ell_2\ 5]}\frac{[\ell_3\ 7]^3}{[7\ 1][1\ \ell_4][\ell_4\ \ell_3]}$$

Multiply and divide by $\langle\ell_4\ \ell_1\rangle^3\langle\ell_2\ \ell_1\rangle^3\langle\ell_1\ \ell_3\rangle^3$.

$$\frac{1}{2}\frac{[\ell_1|\ell_4|\ell_1\rangle^3}{[\ell_1\ 2]\ [2|\ell_4|\ell_1\rangle}\frac{[4|\ell_2|\ell_1\rangle^3}{\langle\ell_1|\ell_2|\ell_1]\ [\ell_1\ 3][3\ 4]}\frac{[5\ 6]^3}{[6|\ell_3|\ell_1\rangle\ \langle\ell_1|\ell_3|\ell_2|\ell_1\rangle\ \langle\ell_1|\ell_2|5]}\frac{\langle\ell_1|\ell_3|7]^3}{[7\ 1]\ [1|\ell_4|\ell_1\rangle\ \langle\ell_1|\ell_4|\ell_3|\ell_1\rangle}$$

Now $\ell_2, \ell_3, \ell_4$ appear as vectors and can be expressed in terms of $\ell_1$.

Expand $|\ell_1\rangle = \alpha|2\rangle + \beta|3\rangle,\ \ |\ell_1] = \gamma|2] + \delta|3]$.

Solve $0 = \ell_2^2 = \ell_3^2 = \ell_4^2$ for $\alpha, \beta, \gamma, \delta$. (Remaining freedom is guaranteed to drop out.)

$$\text{coeff} = -\frac{\langle 1\ 2\rangle^3\langle 2\ 3\rangle^3[5\ 6]^3}{\langle 7\ 1\rangle\langle 3\ 4\rangle\ \langle 2|P_{3,4}|5]\ \langle 2|P_{7,1}|6]\ \langle 2|P_{3,4}P_{5,6}|7\rangle\langle 2|P_{7,1}P_{5,6}|4\rangle}$$

## Double Cut Integrals

A closer look at the phase space integral:

$$C_{i,\ldots,j} = \int d\mu\, A^{\rm tree}(-\ell, i, \ldots, j, \ell{-}K)\, A^{\rm tree}(K{-}\ell, j{+}1, \ldots, i{-}1, \ell)$$

$$d\mu = d^4\ell \;\; \delta^+(\ell^2)\, \delta^+((\ell - K)^2)$$

Cachazo, Svrček, Witten: Change to spinor variables with

$$\ell_{a\dot{a}} = t\, \lambda_a \tilde{\lambda}_{\dot{a}}.$$

$t$ is real. The spinors $\lambda$ and $\tilde{\lambda}$ are independent, homogeneous coordinates on two copies of $CP^1$. The integral is over the contour $\tilde{\lambda} = \bar{\lambda}$.

$$\int d^4\ell\, \delta^+(\ell^2)\, (\bullet) = \int_0^\infty dt\, t \int \langle \lambda\, d\lambda \rangle\, [\tilde{\lambda}\, d\tilde{\lambda}]\, (\bullet)$$

## Steps in spinor integration

(RB, Buchbinder, Cachazo, Feng; RB, Feng, Mastrolia)

- Change variables, $\ell = t\lambda\tilde{\lambda}$, and use the spinor measure,

$$\int d^4\ell \;\, \delta(\ell^2) \;\, \delta((\ell - K)^2) \;\, (\bullet)$$
$$= \int_0^\infty dt\, t \int \langle\lambda\, d\lambda\rangle\, [\tilde{\lambda}\, d\tilde{\lambda}]\, \delta((t\lambda\tilde{\lambda} - K)^2)(\bullet)$$

- Use 2nd delta function to perform $t$-integral.
- Simplify denominators with spinor identities.
- Express result as a total spinor-derivative plus delta functions. May involve introducing one Feynman parameter.

The plan is to use the equation

$$\Delta A_n^{1-\mathrm{loop}} = \sum c\,\Delta I$$

to extract the coefficients and then reconstruct the amplitude.

The discontinuities $\Delta I$ are immediately computable.

First I illustrate the cut of the bubble and 3-mass triangle.

These examples show the essential ideas that we will need for the cut of the amplitude on the left hand side.

## Cutting the bubble

$$\Delta I_2 \quad = \quad \int d^4\ell \;\, \delta^+(\ell^2) \;\, \delta^+((\ell - K)^2)$$

Substitute $\ell = t\lambda\tilde{\lambda}$ and the spinor measure.

Also use:

$$\begin{aligned}
\delta((\ell - K)^2) \quad &= \quad \delta(K^2 - 2K\cdot\ell) \\
&= \quad \delta(K^2 + t\left\langle \lambda|K|\widetilde{\lambda}\right]) \\
&= \quad \frac{1}{\left\langle \lambda|K|\widetilde{\lambda}\right]}\delta\left(t + \frac{K^2}{\left\langle \lambda|K|\widetilde{\lambda}\right]}\right)
\end{aligned}$$

- After $t$-integration we get

$$\Delta I_2 \quad = \quad \int \langle \lambda \, d\lambda \rangle \, [\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{K^2}{\left\langle \lambda | K | \widetilde{\lambda} \right]^2}$$

- A key observation: (Cachazo, Svrček, Witten)

$$[\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{1}{\left\langle \lambda | K | \widetilde{\lambda} \right]^2} = [d\widetilde{\lambda} \, \partial_{\widetilde{\lambda}}] \left( \frac{[\eta \, \widetilde{\lambda}]}{\langle \lambda | K | \eta] \left\langle \lambda | K | \tilde{\lambda} \right]} \right)$$

The integral is naively zero.

However, the contour of integration is where $\widetilde{\lambda}$ is the complex conjugate of $\lambda$, so there is a delta-function contribution because

$$\frac{\partial}{\partial \overline{z}} \frac{1}{(z-b)} = 2\pi \delta(z-b).$$

- Applied to our case, we see that there is a contribution from the pole at

$$|\lambda\rangle = |K|\eta], \qquad \Rightarrow |\widetilde{\lambda}] = |K|\eta\rangle$$

- Final result for the cut bubble:

$$\begin{aligned}
\Delta I_2 &= \int \langle\lambda\, d\lambda\rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}] \frac{K^2}{\left\langle\lambda|K|\widetilde{\lambda}\right]^2} \\
&= -K^2 \left( \frac{[\eta\, \widetilde{\lambda}]}{\left\langle\lambda|K|\widetilde{\lambda}\right]} \right) \Bigg|_{|\lambda\rangle = |K|\eta]} \\
&= -1
\end{aligned}$$

## Cutting the three-mass triangle

$$\Delta I_3 \quad = \quad \int d^4\ell\, \delta^+(\ell^2) \frac{\delta^+((\ell - K_1)^2)}{(\ell + K_3)^2}$$

- After $t$-integration:

$$\Delta I_3 \quad = \quad \int \langle \lambda\, d\lambda \rangle\, [\tilde{\lambda}\, d\tilde{\lambda}] \frac{1}{\left\langle \lambda | K_1 | \tilde{\lambda} \right] \left\langle \lambda | Q | \tilde{\lambda} \right]}$$

where $Q_{a\dot{a}} = \frac{K_3^2}{K_1^2}(K_{1,a\dot{a}}) + (K_{3,a\dot{a}})$ and $Q^2 = \frac{K_3^2 K_2^2}{K_1^2}$.

- Introduce a Feynman parameter:

$$\Delta I_3 \quad = \quad \int_0^1 dz \int \langle \lambda\, d\lambda \rangle\, [\tilde{\lambda}\, d\tilde{\lambda}] \frac{1}{\left\langle \lambda | (1 - z) K_1 + z Q | \tilde{\lambda} \right]^2}$$

- Now we know how to do this integral:

$$\begin{aligned}\Delta I_3 &= -\int_0^1 dz \frac{1}{((1-z)K_1 + zQ)^2} \\ &= -\int_0^1 dz \frac{1}{K_1^2 + 2z((K_1\cdot Q) - K_1^2) + z^2(Q-K_1)^2}\end{aligned}$$

- Define $a = (Q-K_1)^2$, $b = 2((K_1\cdot Q) - K_1^2)$ and $c = K_1^2$. The result is

$$\Delta I_3 = \frac{1}{\sqrt{\Delta_{3m}}} \ln\left(\frac{2az + b - \sqrt{\Delta_{3m}}}{2az + b + \sqrt{\Delta_{3m}}}\right)$$

with

$$\Delta_{3m} = (K_1^2)^2 - 2K_1^2K_2^2 - 2K_3^2K_1^2 + (K_2^2 - K_3^2)^2$$

- The argument of the logarithm (easily identified via the square root $\sqrt{\Delta_{3m}}$) functions as the signature of the three-mass triangle function.

At this point we proceeded by cutting the amplitude and relating the cut integral to the cuts of master integrals.

So far, it has been essential that the unitarity cut is strictly in four dimensions. This is because we relied on the spinor formalism.

The price is that we miss the “rational” terms.

There are neat new ways to get the rational terms of 4-d amplitudes:

- on-shell recursions at one loop (Bern, Dixon, Kosower; Berger, Dixon, Forde, Kosower)
- Feynman diagrams with targeted reduction (Xiao, Yang, Zhu; Binoth, Guillet, Heinrich)

We can get the rational terms as well by working in full dimensional regularization. All one-loop amplitudes are cut-constructible in $4-2\epsilon$ dimensions (van Neerven). We have recently learned how to adapt spinor integration to these calculations in "$d$-dimensional unitarity." (Anastasiou, RB, Feng, Kunszt, Mastrolia)

## Dimensional regularization:

- implicit till now, but computed only up to $\mathcal{O}(\epsilon)$;
- allows complete one-loop calculations, including rational part;
- is the same formalism needed for massive particles.

## Dimensional Regularization

- Write $p = \widetilde{\ell} + \vec{\mu}$, where $\widetilde{\ell}$ is $4$-dimensional and $\vec{\mu}$ is $(-2\epsilon)$-dimensional. (Bern, Chalmers; Bern, Morgan)

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \quad = \quad \int \frac{d^4\widetilde{\ell}}{(2\pi)^4} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} = \int \frac{d^4\widetilde{\ell}}{(2\pi)^4} \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int d\mu^2 (\mu^2)^{-1-\epsilon}.$$

- Relate the $\widetilde{\ell}$ to a null 4-momentum via the cut momentum $K$.

$$\widetilde{\ell} = \ell + zK, \quad \ell^2 = 0$$

$$\int d^4\widetilde{\ell} = \int dz \;\, d^4\ell \;\, (2\ell \cdot K)\, \delta^+(\ell^2).$$

Replace $\mu^2$ by a dimensionless variable $u = \frac{4\mu^2}{K^2} \in [0,1]$.

The $z$-integral is trivial:

$$z = \frac{(1 - \sqrt{1-u})}{2}.$$

# Cutting the Amplitude

$$\int d^4\ell \;\; \delta(\ell^2) \;\; \delta((\ell-K)^2) A_L^{\rm tree} A_R^{\rm tree}$$

$$= \int_0^\infty t\, dt \int_{\widetilde{\lambda}=\lambda} \langle \lambda\, d\lambda \rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}]\, \delta\left(K^2 + t\left\langle \lambda|K|\widetilde{\lambda}\right]\right) A_L^{tree}(t,\lambda,\widetilde{\lambda}) A_R^{tree}(t,\lambda,\widetilde{\lambda})$$

- After $t$-integration, we find terms of the form:

$$I_{term} = \frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j \; \tilde{\lambda}]}{\left\langle \lambda|K|\tilde{\lambda}\right]^n \prod_{i=1}^{k} \left\langle \lambda|Q_i|\tilde{\lambda}\right]}$$

- Apply Schouten's identity to split up denominators; express as total derivatives plus delta functions.
- Identify expressions with cuts of basis integrals and read off coefficients.
- We have given formulas for the coefficients and instructions for how to evaluate them. (RB, Feng)

## Box coefficients

$$C_{box;ij} \quad = \quad \frac{K^2}{\sqrt{1-u}} \left( \frac{F_{i,j}(P_1^{(ij)}) + F_{i,j}(P_2^{(ij)})}{2} \right)$$

$$P_{1,2}^{(ij)} \quad = \quad Q_j + y_{1,2}^{(ij)} Q_i \quad (i < j)$$

$$y_{1,2}^{(ij)} \quad = \quad \frac{-2Q_i \cdot Q_j \pm \sqrt{\Delta^{(ij)}}}{2Q_i^2}, \quad \Delta^{(ij)} = (2Q_i \cdot Q_j)^2 - 4Q_i^2 Q_j^2$$

$$F_{i,j}(\ell) = \frac{G(\lambda) \prod_{s=1}^{n+k-2} \left[ a_s | Q_i | \ell \right\rangle}{\langle \ell | K Q_i | \ell \rangle^n \prod_{t=1, t \neq i,j}^{k} \langle \ell | Q_t Q_i | \ell \rangle}$$

## Boxes and Pentagons

In $d$ dimensions, pentagons are independent master integrals. But the cut pentagons are closely related to boxes. (In pentagon integrands, powers of $u$ can appear in the denominator.)

The preceding formula actually included both box and pentagon coefficients.

We have to do polynomial division:

$$C_{box;ij}(u) = H(u) + \sum_l A_l P_l$$

$H(u)$ is polynomial; $P_l$ the pentagon cut; $A_l$ is constant in $u$.

Then $H(u)$ is the box coefficient and $A_l$ is the pentagon coefficient.

# Triangle coefficients

$$C_{tri;i} = \frac{\sqrt{\Delta^{(i)}}}{2\sqrt{1-u}} \left\{ \left( \frac{G(\lambda) \prod_{s=1}^{n+k-2} \left[a_s|Q_i|\lambda\right\rangle}{\langle\lambda|KQ_i|\lambda\rangle^n \prod_{t=1,t\neq i}^{k} \langle\lambda|Q_tQ_i|\lambda\rangle} \right)\Bigg|_{\langle\lambda\, P_1^{(i)}\rangle=0} - \left( \frac{G(\lambda) \prod_{s=1}^{n+k-2} \left[a_s|Q_i|\lambda\right\rangle}{\langle\lambda|KQ_i|\lambda\rangle^n \prod_{t=1,t\neq i}^{k} \langle\lambda|Q_tQ_i|\lambda\rangle} \right)\Bigg|_{\langle\lambda\, P_2^{(i)}\rangle=0} \right\}$$

$$P_{1,2}^{(i)} = Q_i + x_{1,2}^{(i)} K$$

$$x_{1,2}^{(i)} = \frac{-2Q_i \cdot K \pm \sqrt{\Delta^{(i)}}}{2K^2}, \quad \Delta^{(i)} = (2Q_i \cdot K)^2 - 4Q_i^2 K^2$$

## Residues from multiple poles

Could evaluate by splitting with Schouten identities. (RB, Feng, Mastrolia) More directly, introduce a parameter to break degeneracy:

$$\frac{1}{\langle \ell\, \eta\rangle^n}\frac{\prod_i \langle \ell\, a_i\rangle}{\prod_j \langle \ell\, b_j\rangle} \rightarrow \frac{1}{\prod_{s=0}^{n-1}\langle \ell\,(\eta+s\tau\alpha)\rangle}\frac{\prod_i \langle \ell\, a_i\rangle}{\prod_j \langle \ell\, b_j\rangle}$$

$$= \frac{1}{\tau^{n-1}\langle \eta\,\alpha\rangle^{n-1}}\frac{\prod_i \langle \eta\, a_i\rangle}{\prod_j \langle \eta\, b_j\rangle}\left(\sum_{s_0=0}^{n-1}\frac{1}{\prod_{s=0,s\neq s_0}^{n-1}(s-s_0)}\frac{\prod_i (1+\tau s_0\frac{\langle \alpha\, a_i\rangle}{\langle \eta\, a_i\rangle})}{\prod_j (1+\tau s_0\frac{\langle \alpha\, b_j\rangle}{\langle \eta\, b_j\rangle})}\right).$$

Expand expression inside parentheses in $\tau$, up to order $\tau^{n-1}$. Result for residue:

$$\frac{1}{\langle \eta\,\alpha\rangle^{n-1}}\frac{\prod_i \langle \eta\, a_i\rangle}{\prod_j \langle \eta\, b_j\rangle}\left(\sum_{s_0=0}^{n-1}\frac{s_0^{n-1}}{\prod_{s=0,s\neq s_0}^{n-1}(s-s_0)}\sum\left(\prod_{j=1}^{k}(-)^{m_j}\left(\frac{\langle \alpha\, b_j\rangle}{\langle \eta\, b_j\rangle}\right)^{m_j}\right)\right.$$
$$\left.\left(\sum\prod_{q=1}^{N_1}\frac{\langle \alpha\, a_{i_q}\rangle}{\langle \eta\, a_{i_q}\rangle}\right)\right).$$

(RB, Feng)

## Bubble coefficients

$$\text{Rational} = \frac{1}{\sqrt{1-u}} \sum_{s=1}^{n-1} R[\widetilde{K}(s), \{Q_i\}, \eta]\Big|_{\tau \to 0}$$

The limit is taken by expanding and truncating the series.

$$R[\widetilde{K}(s), \{Q_i\}, \eta] =$$

$$\sum_{\text{poles except } \eta} \text{Res} \left( \frac{G(\lambda) \prod_{j=1}^{n+k-2} \left[ a_j | \widetilde{K}(s) | \lambda \right\rangle}{\tau^{n-2} \langle \lambda | \eta \widetilde{K}(s) | \lambda \rangle^{n-1} \prod_{s'=1, s' \neq s}^{n-1} (s' - s) \prod_{i=1}^{k} \langle \lambda | Q_i \widetilde{K}(s) | \lambda \rangle} \right.$$

$$\left. \times \left( \sum_{h=0}^{n-2} \frac{\tau^h s^h \left\langle \lambda | \eta | \tilde{\lambda} \right]^{h+1}}{(h+1) \left\langle \lambda | \widetilde{K}(s) | \tilde{\lambda} \right]^{h+1}} \right) \right)$$

$$\widetilde{K}(s) = K + \tau s \eta$$

The phase space measure becomes (Anastasiou, RB, Feng, Kunszt, Mastrolia)

$$\int du\, u^{-1-\epsilon} \int \langle \lambda\, d\lambda \rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}] \int_0^\infty dt\, t\, \delta\left(\sqrt{1-u}K^2 + t\left\langle \lambda | K | \widetilde{\lambda} \right]\right)$$

We recognize and perform the spinor integral as before. Any 4-dimensional technique can be applied.

For physical reasons (PV reduction), the integrand must be a sum of the following (after extracting pentagons):

$$\begin{aligned}
\mathrm{Bub}^{(n)} &= \int_0^1 du\, u^{-1-\epsilon} u^n \sqrt{1-u} \\
\mathrm{Tri}^{(n)} &= \int_0^1 du\, u^{-1-\epsilon} u^n \ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right) \\
\mathrm{Box}^{(n)} &= \int_0^1 du\, u^{-1-\epsilon} \frac{u^n}{\sqrt{B-Au}} \ln\left(\frac{D-Cu-\sqrt{1-u}\sqrt{B-Au}}{D-Cu+\sqrt{1-u}\sqrt{B-Au}}\right)
\end{aligned}$$

where $Z^2, A, B, C, D$ are rational functions of kinematical invariants of the external momenta.

## Dimensional shift identities

$$\mathrm{Bub}^{(n)} = F^{(n)}_{2\to 2}\mathrm{Bub}^{(0)}$$
$$\mathrm{Tri}^{(n)} = F^{(n)}_{3\to 3}\mathrm{Tri}^{(0)} + F^{(n)}_{3\to 2}\mathrm{Bub}^{(0)}$$
$$\mathrm{Box}^{(n)} = F^{(n)}_{4\to 4}\mathrm{Box}^{(0)} + F^{(n)}_{4\to 3}\mathrm{Tri}^{(0)} + F^{(n)}_{4\to 2}\mathrm{Bub}^{(0)}$$

$$F^{(n)}_{2\to 2} = \frac{(-\epsilon)_{\frac{3}{2}}}{(n-\epsilon)_{\frac{3}{2}}}, \quad F^{(n)}_{3\to 3} = \frac{-\epsilon}{n-\epsilon}(1-Z^2)^n,$$

$$F^{(n)}_{4\to 4} = \frac{(-\epsilon)_{\frac{1}{2}}}{(n-\epsilon)_{\frac{1}{2}}}\left(\frac{B}{A}\right)^n,$$

$$F^{(n)}_{3\to 2} = \frac{(-\epsilon)_{\frac{3}{2}}}{n-\epsilon}\sum_{k=1}^{n}\frac{2Z(1-Z^2)^{n-k}}{(k-\epsilon)_{\frac{1}{2}}}$$

$$F^{(n)}_{4\to j} = \frac{D+(Z^2-1)C}{(n-\epsilon)_{\frac{1}{2}}ZA}\sum_{k=1}^{n}\left(\frac{B}{A}\right)^{n-k}\frac{F^{(k-1)}_{3\to j}}{(k-1/2-\epsilon)_{\frac{1}{2}}}$$

We have also given reduction formulas with explicit propagator masses.

(RB, Feng)

In this case, the basis also includes tadpoles. These cannot be detected by unitarity methods. We believe these coefficients can be determined by divergences or a heavy mass limit.

## Summary

- Spinors are a good choice for writing massless particles in 4 dimensions. With spinor integration techniques, we have advanced the unitarity approach to computing amplitudes.
- Quadruple cuts isolate scalar box coefficients.
- Isolate coefficients of any master integrals by matching logarithms in discontinuities. Algebraic$^*$ operations. Formulas given.
- Dimensional shift identities for dimensionally regularized integrals. Results to all orders in $\epsilon$.