Analytic Techniques for Massive One-Loop Amplitudes

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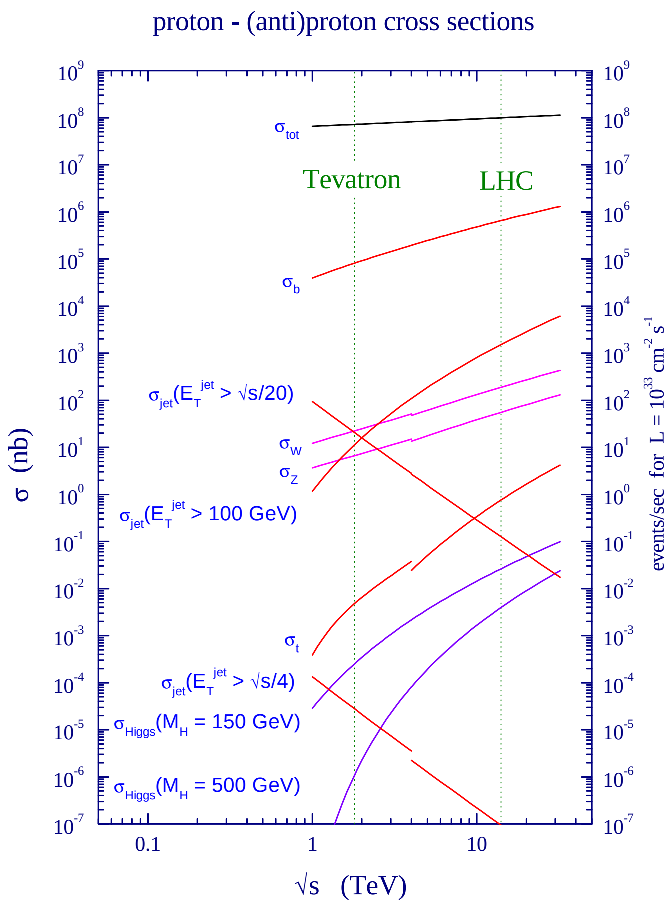
Fermilab Theory Seminar

Outline

- Mutli-leg one-loop amplitudes and on-shell simplifications
- Review of on-shell (and not so on-shell) methods
  - Complex analysis and generalised unitarity
  - Feynman reduction with higher dimensional integrals
- New methods for un-ordered amplitudes
  - Hidden symmetries: No-triangles in QED
  - Compact QED tree-amplitudes
- Application to massive fermion amplitudes
  - Spinor/Helicity with massive fermions
  - Constraints from universal UV/IR poles
  - Compact helicity amplitudes for $p\bar{p} \to t\bar{t}$

NLO Computations

- New physics searches @ Tevatron and LHC
  - Massive states
  - Many jets
- General signals hidden by complicated QCD backgrounds
  - NLO required for quantitative predictions
  - Fast and accurate differential cross sections
  - Resummations and Parton showers



Computations of Virtual Corrections

- A lot of recent progress in computational methods for virtual corrections:

  Bern,Dixon,Dunbar,Kosower,Britto,Cachazo,Feng,Mastrolia,
  Ossola,Papadopoulos,Pittau,Ellis,Giele,Kunszt,Melnikov,Forde,...

- Automated numerical approaches:

  [`BlackHat`, `Rocket`, `CutTools/Helac-1loop`, `GOLEM`, Denner et al.,`samurai`, ...]

  - Growing number of phenomenological studies
  - Continuing improvements: colour dressing, GPU's

  [Giele,Kunszt,Winter][Giele,Stavenga,Winter]

- Efficiency:
  - Numerical stablility
  - Fast numerical evaluation
  - Complexity of processes with additional jets
  - Portability $\Rightarrow$ Public codes
- Analytical computations good for numerical stability and speed
- Amplitude structures can give insight into new techniques

Recent Highlights

- Numerical:
  - $gg \to 20(g)$ [Giele,Zanderghi]
  - $p\bar{p} \to W/Z + 3j$ [Berger et al. `Blackhat`] [Ellis,Giele,Kunszt,Melnikov,Zanderighi `Rocket`]
  - $p\bar{p} \to t\bar{t} + 2j$ [Bevilaqua et al. `Helac-1l`]
  - $p\bar{p} \to t\bar{t} + b\bar{b}$ [Bredenstein,Denner,Dittmaier,Pozzorini] [Bevilaqua et al. `Helac-1l`]
  - $p\bar{p} \to WW + 2j$ [Melia,Melinkov,Rontsch,Zanderighi]
  - $e^+e^- \to 5j$ [Frederix,Frixione,Melinkov,Zanderighi]
  - plus many more...
- Analytic: (including development of methods)
  - $gg \to gggg$

    Bern,Dixon,Kosower,Dunbar,Forde,Berger,Britto,Feng,Cachazo,Mastrolia,...
  - $p\bar{p} \to H + 2j$

    SB,Glover,Risager,Williams,Mastrolia,Berger, Dixon,Del Duca,Sofianatos,Campbell,Ellis

    Cross section with decays in MCFM [Ellis,Campbell,Williams (2010)]

Spinor-Helicity Formalism

- All amplitudes described by two-component Weyl spinors

$$\langle pq \rangle = e^{i\theta_{pq}} \sqrt{p \cdot q} \qquad [pq] = e^{-i\theta_{pq}} \sqrt{p \cdot q}$$

- Momenta and polarisation vectors

$$p^\mu = \frac{1}{2} \langle p|\sigma^\mu|p] \qquad \varepsilon_+^\mu(p,\xi) = \frac{\langle \xi|\sigma^\mu|p]}{\sqrt{2}\langle \xi p \rangle} \qquad \varepsilon_-^\mu(p,\xi) = \frac{\langle p|\sigma^\mu|\xi]}{\sqrt{2}[p\xi]}$$

- Massive momenta $\Rightarrow$ longer “spinor strings"

$$\langle p|P|q] \qquad \langle p|PQ|q\rangle \qquad \langle p|P(p+q)|r\rangle$$

Simple structures at tree level

- Huge on-shell cancellations in tree-level QCD
  c.f. factorial growth of diagrams
- Example: $gg \to n(g)$ [Parke,Taylor (1985)]

$$A_n^{(0)}(1^+, 2^+, \ldots, n^+) = 0 \qquad \text{"all-plus"}$$

$$A_n^{(0)}(1^-, 2^+, \ldots, n^+) = 0 \qquad \text{"one-minus"}$$

$$A_n^{(0)}(1^-, 2^-, \ldots, n^+) = \frac{\langle 12 \rangle^4}{\prod_{k=1}^{n} \langle k(k+1) \rangle} \qquad \text{MHV}$$

- Also in QED $q\bar{q} \to n(\gamma)$ [Kleiss,Stirling (1986)]

$$A_n^{(0)}(n_{\bar{q}}^+, 1_q^-, 2^-, \ldots, (n-1)^+) = \frac{\langle 12 \rangle^3 \langle n2 \rangle \langle 1n \rangle^{n-2}}{\prod_{k=2}^{n-1} \langle 1k \rangle \langle nk \rangle} \qquad \text{MHV}$$

Colour Ordering and Primitive Amplitudes

- Order kinematic dependence w.r.t. group structure :
  Colour Ordered / Partial Amplitudes

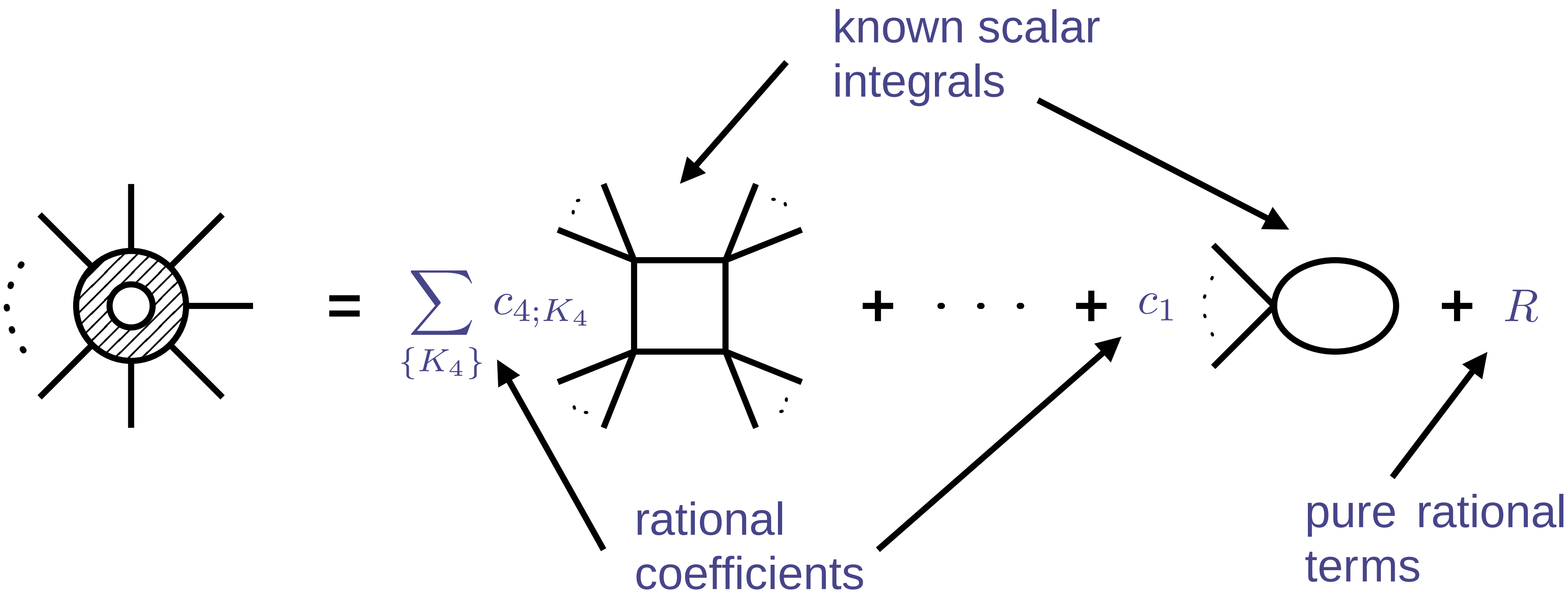
$$\mathcal{A}^{(0)}(\{a_i\},\{h_i\},\{p_i\}) = \sum_c f_c(T^{a_i}) A^{(0)}(\{h_i\},\{p_i\})$$

- At higher loops internal colour flows give additional structure :
  Primitive Amplitudes

$$A^{(l)}(N_c, N_f, \{h_i\},\{p_i\}) = \sum_p g_p(N_c, N_f) A^{(l),[p]}(\{h_i\},\{p_i\})$$

- Primitive amplitudes are minimal gauge invariant sub-sets

Structure of One-Loop Amplitudes

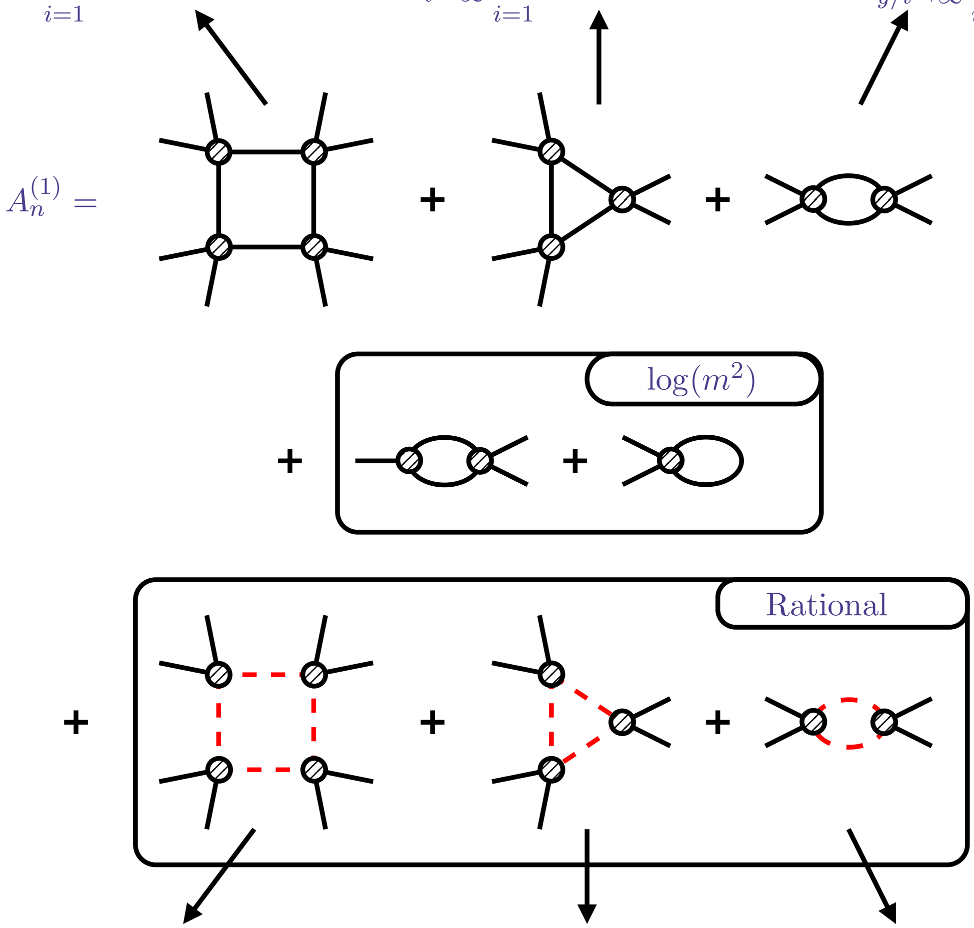


- General gauge theory amplitudes reduced to box topologies or simpler

  [Passarino,Veltman;Melrose]
- Isolate logarithms with cuts $\rightarrow$ exploit on-shell simplifications
- General cutting principle:
  - apply δ-functions to L and R sides
  - generate and solve the linear system for the coefficients

Generalised Unitarity for One-Loop Amplitudes

$$C_4 = \prod_{i=1}^{4} A_i(l) \qquad C_3 = \lim_{t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0} \qquad C_2 \sim \lim_{y/t\to\infty} \prod_{i=1}^{2} A_i(l(y,t))$$



$$C_4^{[4]} = \lim_{\mu^2\to\infty} \prod_{i=1}^{4} A_i(l)|_{\mu^4} \qquad C_3^{[2]} = \lim_{\mu^2,t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0,\mu^2} \qquad C_2^{[2]} \sim \lim_{\mu^2,y/t\to\infty} \prod_{i=1}^{2} A_i(l(y,t))$$

Generalised cuts $\Rightarrow$ loops from trees

Bern,Dixon,Dunbar,Kosower;

Britto,Cachazo,Feng,Mastrolia,Yang;

Ossola,Papadopoulos,Pittau;

Forde;Kilgore

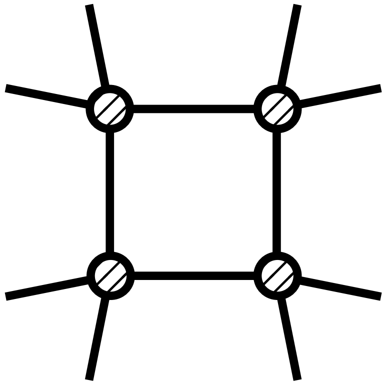
D-dimensional cuts:

[Ossola,Papadopoulos,Pittau;

Giele,Kunszt,Melnikov;

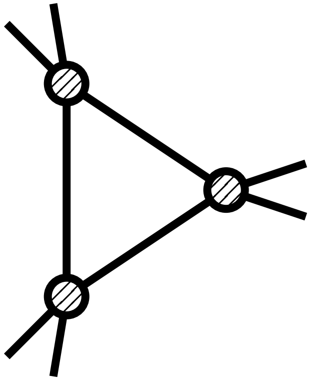
Britto,Feng,Mastrolia;SB]

Multiple Cuts and Integrand Reduction



- Quadruple cut $\rightarrow$ 4 on-shell δ-functions
- $l_k^2 = m_k^2 \rightarrow$ fixed loop momentum
- $C_4 = \frac{1}{2}\sum_{\sigma=\pm} A_1 A_2 A_3 A_4(l_1^\sigma)$ [BCF]

- Triple cut $\rightarrow$ 3 on-shell δ-functions
- Parametrise free integration [OPP,Forde]



$$\oint J_t dt A_1 A_2 A_3 = \oint J_t dt \mathrm{Inf}_t[A_1 A_2 A_3(t)] + \sum_k \frac{\mathrm{Res}_{\mathrm{t}=\mathrm{t_k}}(A_1 A_2 A_3)}{\xi_k (t - t_k)}$$

- $C_3 = \frac{1}{2}\sum_{\sigma=\pm} \mathrm{Inf}_t[A_1 A_2 A_3(l_1^\sigma(t))]|_{t^0}$

Bubble coefficients follow from a similar analysis:

$$C_2 = \mathrm{Inf}_t \mathrm{Inf}_y[A_1 A_2(t,y)] - \frac{1}{2}\sum_{\sigma=\pm} \mathrm{Inf}_t[A_1 A_2' A_3'(t, y_\pm)]$$

3-cut: Cauchy's Theorem [Dunbar,Perkins,Warwick]

2-cut: Stokes' Theorem [Mastrolia]

Computation of the Rational terms

- D-dimensional cuts perfectly valid... [Ellis,Giele,Kunszt,Melnikov]
  [Anastasiou,Britto,Feng,Kunszt,Mastrolia][SB]
- On-shell recursion relations [Berger,Bern,Dixon,Forde,Kosower]
- Feynman methods may offer a more direct route to compact analytic forms for the rational terms. $[pp \to H + 2j]$
- Provide important cross checks for newer on-shell techniques
- Bottleneck: Passerino-Veltamn reduction produces large expressions with unphysical singularities
- Reduction to using higher dimensional integrals offers a number of advantages: [Davydychev (1991)]
  - Direct match with D-dimensional cut basis
  - Explicit cancellation of some Gram determinants

[Diakonidis,Fleischer,Riemann,Tausk][Fleischer,Riemann]

Tensor Reduction for Feynman Diagrams

- Full numerical reduction program in c++ [Yundin(in preparation)]

  (1) Tensor reduction to higher dimension integrals

$$I_n^{\mu_1\dots\mu_n} = \sum_{m=1}^{5} \sum_{K_m} \sum_{l<m} T^{\mu_1\dots\mu_n}_{m,l|K_m} I^{4+2l-2\epsilon}_{m|K_m}$$

  (2) Reduce to scalar integrals using dimension shift

$$I_m^{D+2} = \frac{A}{G} I_m^D + \sum_k \frac{B_k}{G} I^D_{m-1;k}$$

[Bern,Dixon,Kosower]

[Fleischer,Jegerlehner,Tarasov][Tarasov]

  (2b) Elliminate Gram determinants, G, in step (2) [Fleischer,Riemann]

- Analytic implementation for rational terms in $p\bar{p} \to t\bar{t}$

QED trees and on-shell recursion

Simplicity in “unordered" theories

- QED and Gravity amplitudes contain simplicity not seen in QCD
  - Improved multi-loop UV behaviour [Bern,Carrasco,Dixon,Johnasson,Roiban]
  - No-triangles in $N=8$ super-gravity [Arkani-Hamed,Cachazo,Kaplan] [Bjerrum-Bohr,Vanhove]
  - Amplitude relations : KLT, BCJ, Monodromies etc. [Kawai,Lewellen,Tye] [Bern,Carrasco,Johnasson] [Bjerrum-Bohr,Damgaard,Feng,Sondergaard]
  - No-triangles in n-photon amplitudes for $n>6$ [SB,Bjerrum-Bohr,Vanhove]

    proof via generalised unitarity with massive cuts
- Study UV properties via large momentum limits with on-shell recursion

Review of BCFW recursion

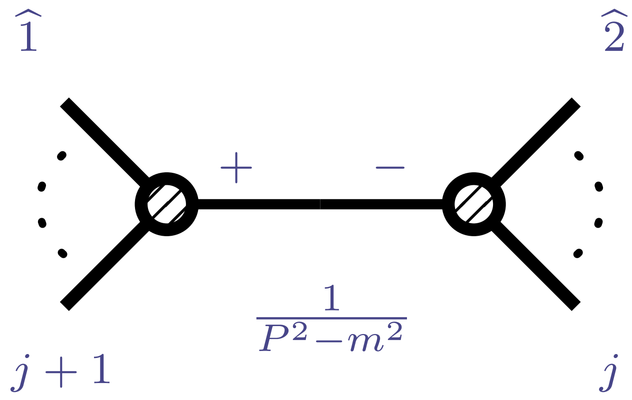
[Britto,Cachazo,Feng,Witten (2004)]

- Tree amplitudes constructed using Cauchy theorem:

$$\oint \frac{A(z)}{z} dz$$

$$A(z) \stackrel{z\to\infty}{\to} 0 \Rightarrow A(0) = -\sum_{\text{poles }k} A_L(z_k)\frac{1}{P_k^2}A_R(z_k)$$

$$\sum_{j=3}^{n-1}$$



- In most cases gives shortest known representations

Dressing the Recursion Relation

- For QED and Gravity it does not. . .
  - Gravity : $M_n(z) \stackrel{z\to\infty}{\to} \frac{1}{z^2}$
  - $q\bar{q} + n(\gamma)$: $A_n(z) \stackrel{z\to\infty}{\to} \frac{1}{z^{n-2}}$
- Improved UV scaling allows modification of BCFW [SB,Henn (2010)]

$$\oint \frac{f(z)A(z)}{z} dz = 0$$

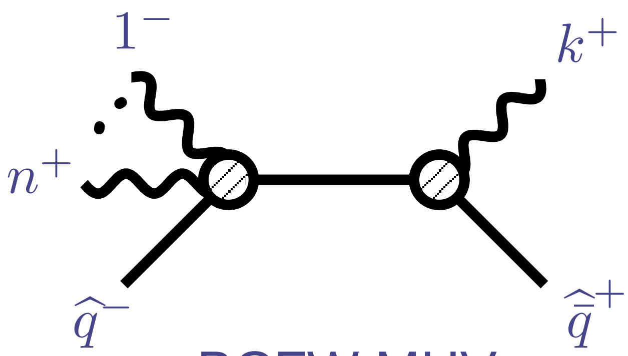
- $f(z)$ can take the form of poles in $A(z) \Rightarrow$ reduction in number of residues

$$A(0) = -\sum_{\text{poles } k} A_L(z_k)\frac{\mathcal{F}(z_k)}{P_k^2}A_R(z_k)$$

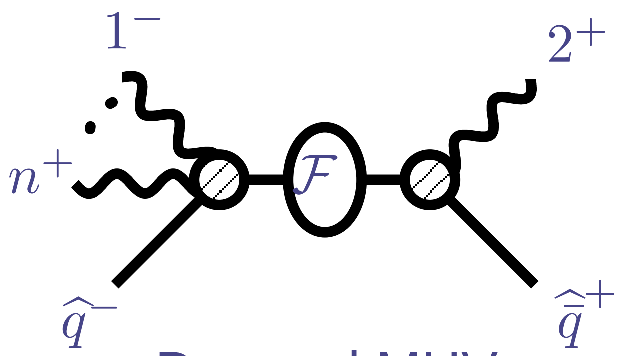
Compact QED Tree amplitudes

- MHV amplitude:

$$\sum_{k=3}^{n}$$



BCFW MHV



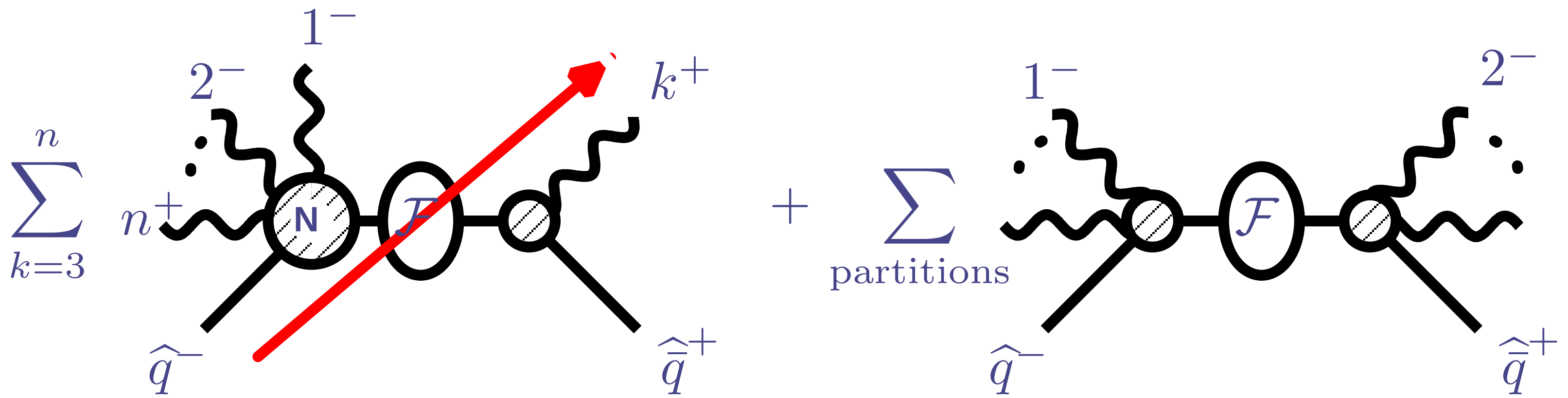
Dressed MHV

- Simple derivation of Kleiss-Stirling formula

$$\mathcal{F}(z) = \prod_{k=3}^{n} \frac{z_2 - z_k}{z_k} = \frac{\langle q\bar{q}\rangle^{n-2}}{\langle q2\rangle^{n-2}} \prod_{k=3}^{n} \frac{\langle 2k\rangle}{\langle \bar{q}k\rangle}$$

$$A_{n;q}^{\mathrm{tree}}(q^-, \bar{q}^+; 1^-, 2^+, \ldots, n^+) = i \frac{\langle q1\rangle^2}{\langle q\bar{q}\rangle} \prod_{k=2}^{n} \frac{\langle q\bar{q}\rangle}{\langle qk\rangle\langle \bar{q}k\rangle} .$$

Compact QED Tree amplitudes



- NMHV and N^{2}MHV : exponential growth in number of terms

n	1	2	3	4	5	6	7	8	9
NMHV,BCFW	1	5	22	103	546	3339	23500	188255	1694806
NMHV,dressed	2	6	14	30	62	126	254	510	1022

- c.f. naiive $\sim n!$, QCD $\sim (n-2)!$ (Kleiss-Kuijf), Gravity $\sim (n-3)!$

One-Loop Helicity Amplitudes for $p\bar{p} \to t\bar{t}$

$t\bar{t}$ production at the LHC

- LHC is a top factory! ($> 10^6$ events/year)
- $pp \to t\bar{t}j$ and $pp \to t\bar{t}b\bar{b}$: backgrounds for $t\bar{t}H$ production channel

  [Dittmaier,Uwer,Weinzierl (2006)]

  [Melnikov,Schulze (2010)]

  [Bredenstein,Denner,Dittmaier,Pozzorini; Bevilaqua et al.`Helac-1loop` (2010)]
- Recent computation of $pp \to t\bar{t} + 2j$ [Bevilacqua et al. `Helac-1loop` (2010)]
- $t\bar{t} + n(j)$ backgrounds for SUSY and LED searches
- $p\bar{p} \to t\bar{t}$ is a well known process:
  - $A^{(1)}A^{(0)}$ Inteference [Ellis,Nason,Dawson (1988)]
  - Tensorial Amplitude : [Körner,Merebashvili (2002)]

    [available in MCFM; Ellis,Campbell (2010)]
  - Spin-Density Matrix : [Bernreuther,Brandenburger,Si,Uwer (2004)]
  - Numerical Unitarity : [Melnikov,Schulze (2009)]

Heavy quark helicity states

- Define heavy quark helicity states w.r.t massless direction η [Kleiss,Stirling]

$$u_\pm(Q;\eta) = \frac{(\not{Q} + m)|\eta\mp\rangle}{\langle Q^\flat \pm |\eta\mp\rangle}$$

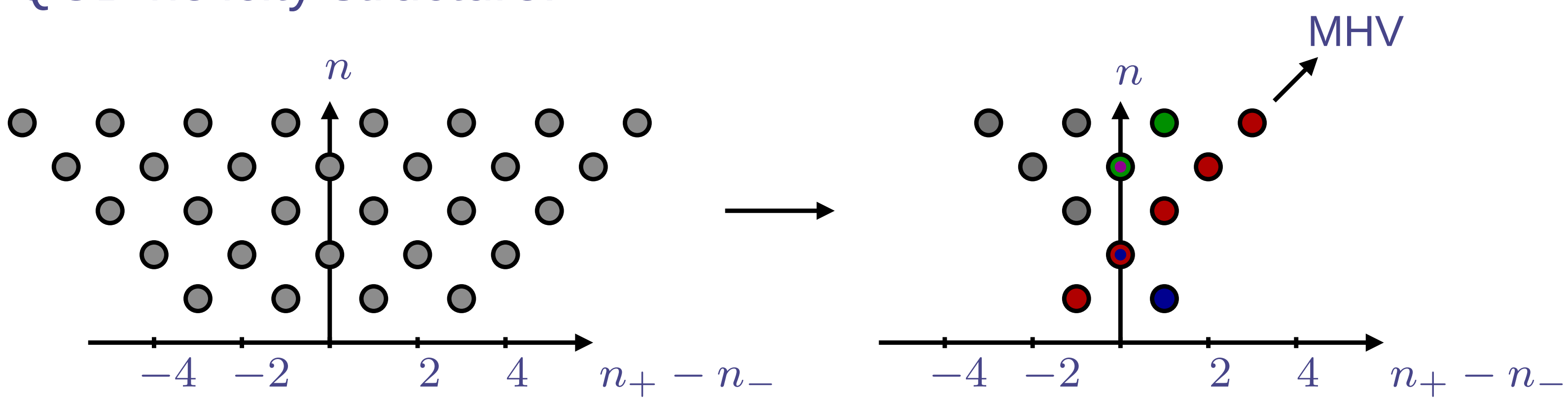
  where

$$Q^{\flat,\mu} = Q^\mu - \frac{m^2}{2Q\cdot\eta}\eta^\mu$$

- Specific choice of reference momentum maps directly to standard dirac representation
- Working with arbitrary reference vectors
  $\Rightarrow$ more independent spinor products
  $\Rightarrow$ less independent helicity amplitudes
- Implementing (LO) decays straightforward

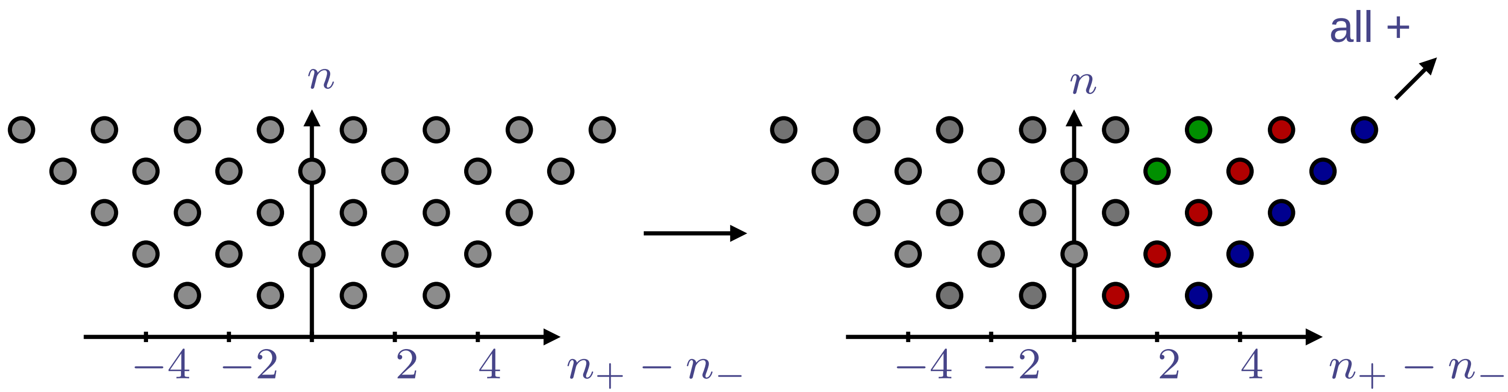
Massive Helicity Amplitudes

Massless QCD helicity structure:



Symmetry relates massive quark helicity states:

$$u_-(p, m; p^\flat, \eta) = \frac{\langle p^\flat \eta \rangle}{m} u_+(p, m; \eta, p^\flat)$$



$gg \to t\bar{t}$ Colour Ordering

- Decompose into primitive amplitudes [Bern,Dixon,Kosower (1994)]

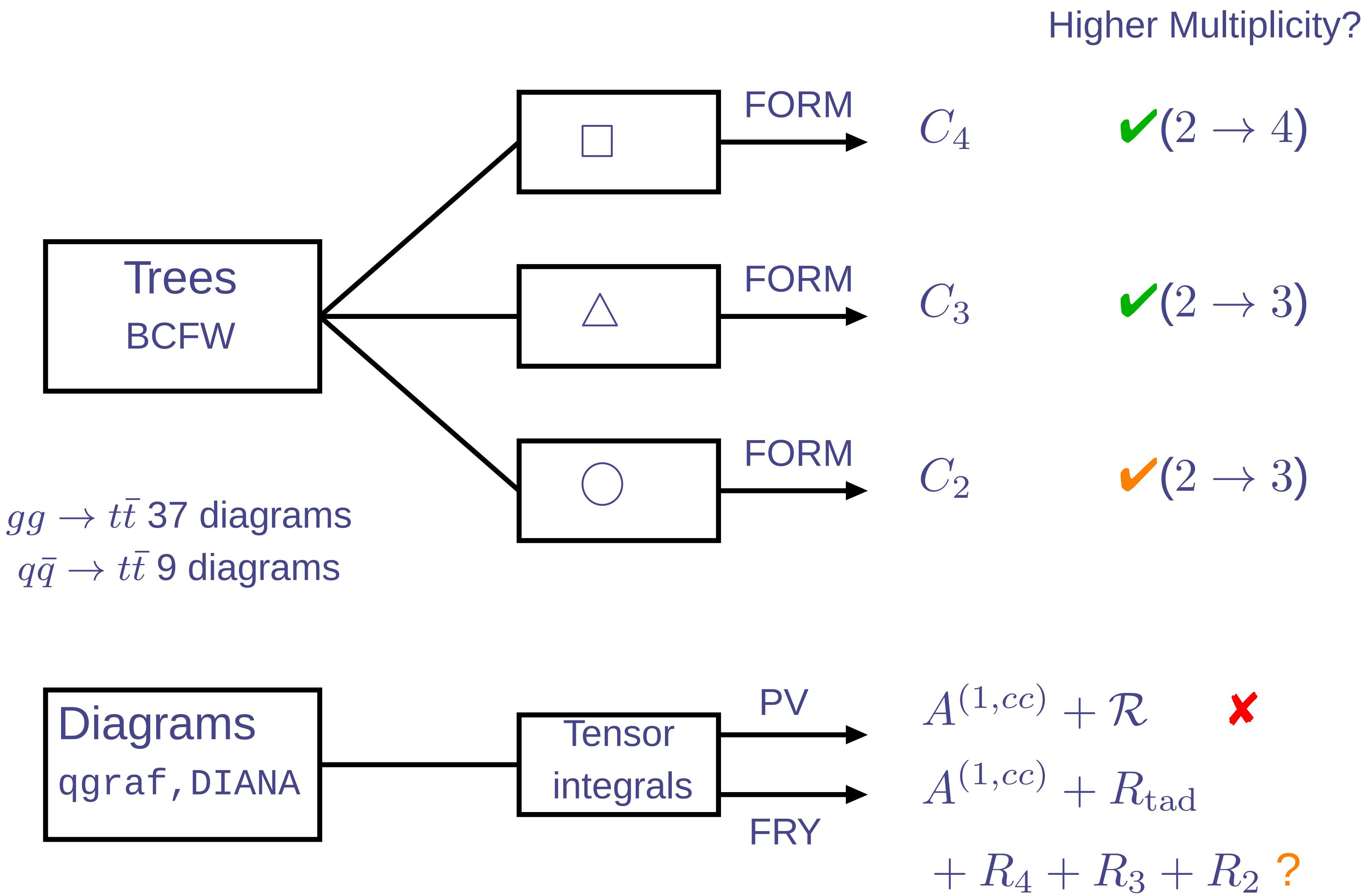
$$\mathcal{A}_4^{(0)}(1_t, 2, 3, 4_{\bar{t}}) = \sum_{P(2,3)} (T^{a_2} T^{a_3})_{i_1 i_4} A_4^{(0)}(1_t, 2, 3, 4_{\bar{t}})$$

$$\mathcal{A}_4^{(1)}(1_t, 2, 3, 4_{\bar{t}}) = \sum_{P(2,3)} N (T^{a_2} T^{a_3})_{i_1 i_4} A_{4;1}^{(1)}(1_t, 2, 3, 4_{\bar{t}}) + \delta^{a_2 a_3} \delta_{i_1 i_4} A_{4;3}^{(1)}(1_t, 4_{\bar{t}}; 2, 3)$$

where

$$\begin{aligned} A_{4;1}^{(1)}(1_t, 2, 3, 4_{\bar{t}}) &= A^{[L]}(1_t, 2, 3, 4_{\bar{t}}) - \frac{1}{N^2} A^{[R]}(1_t, 2, 3, 4_{\bar{t}}) \\ &\quad + \frac{N_f}{N} A^{[f]}(1_t, 2, 3, 4_{\bar{t}}) + \frac{N_H}{N} A^{[H]}(1_t, 2, 3, 4_{\bar{t}}) \\ A_{4;3}^{(1)}(1_t, 4_{\bar{t}}; 2, 3) &= \sum_{P(2,3)} \left\{ A^{[L]}(1_t, 2, 3, 4_{\bar{t}}) + A^{[L]}(1_t, 2, 4_{\bar{t}}, 3) + A^{[R]}(1_t, 2, 3, 4_{\bar{t}}) \right\}. \end{aligned}$$

Computational Strategy



Tree Level Amplitudes

- Compact tree-level amplitudes from BCFW recursion

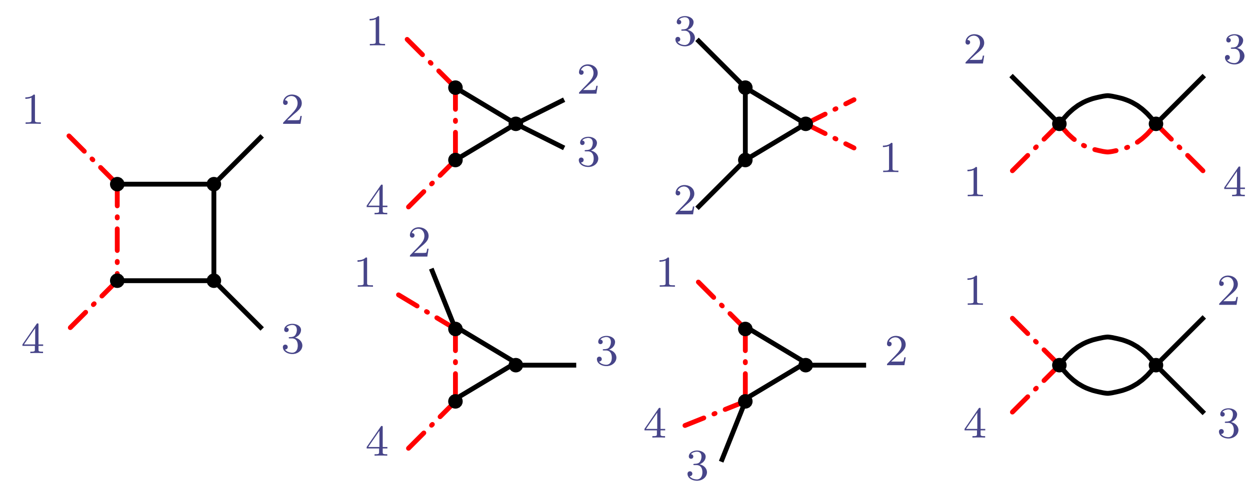
$$A_4(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) = -im^3 \frac{[23]\langle\eta_1\eta_4\rangle}{\langle 23\rangle\langle 2|1|2]\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle}$$

$$A_4(1_t^+, 2^+, 3^-, 4_{\bar{t}}^+) = -im \frac{\langle 3|1|2]\left(\langle\eta_1\eta_4\rangle\langle 3|1|2] + [23]\langle\eta_1 3\rangle\langle\eta_4 3\rangle\right)}{s_{23}\langle 2|1|2]\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle}$$

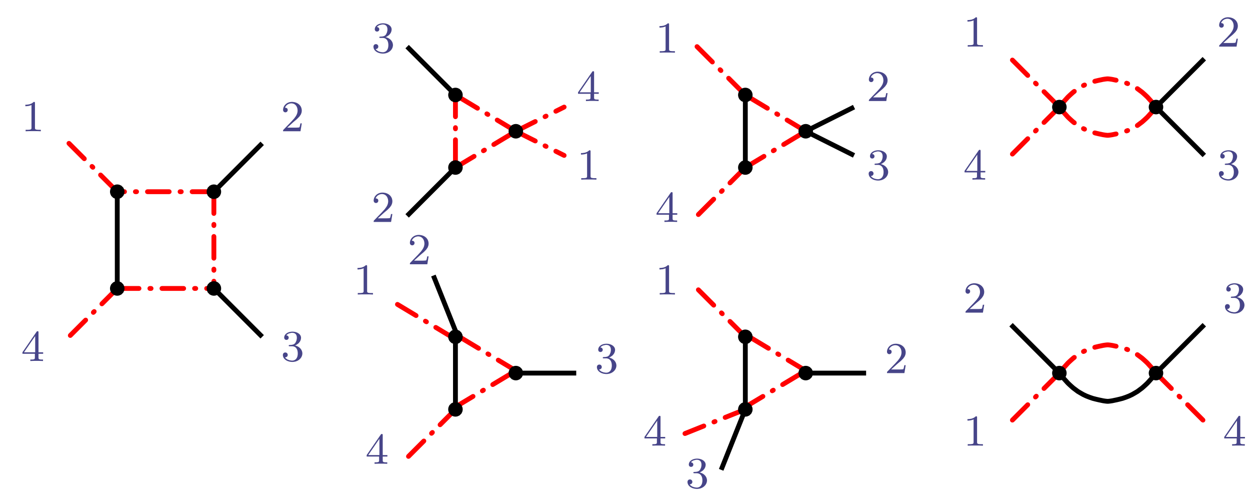
- Easy to automate analytically: All helicity amplitudes for $n \leq 6$
- Turn trees into loops via generalised unitarity
  - Compact expressions at one-loop?
- Study $gg \to t\bar{t}$ in detail. . .

The Integral Basis

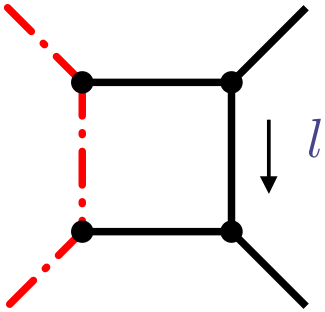
$$A^{[L]}(1_t, 2, 3, 4_{\bar{t}})$$



$$A^{[R]}(1_t, 2, 3, 4_{\bar{t}})$$



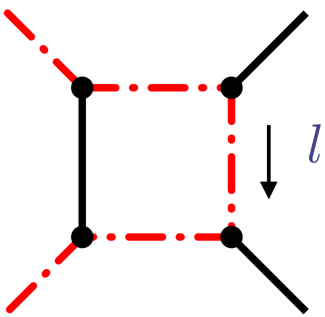
Extracting the Coefficients



$$l^{(1),\mu} = -\frac{\langle 2|1|2]}{2\langle 2|1|3]}\langle 2|\gamma^\mu|3]$$

$$l^{(2),\mu} = -\frac{\langle 2|1|2]}{2\langle 3|1|2]}\langle 3|\gamma^\mu|2]$$

“Left-Moving" box coefficient has simple solution



$$l^{\pm\mu} = \frac{c_\pm}{2}\langle 2|\gamma^\mu|3] - \frac{m^2}{2c_\pm}\langle 3|\gamma^\mu|2]$$

$$c\pm = -\frac{1}{2\langle 2|1|3]}\Big(s_{12} + m^2 \pm \sqrt{(s_{12}+m^2)^2 + 4m^2\langle 2|1|3]\langle 3|1|2]/s_{23}}\Big)$$

“Right-Moving" box coefficient contains square roots. . .

Using the universal IR constraints

- Taylor expansions can result in “relatively" long expressions
  $\Rightarrow$ Further simplifications needed
- Coefficients satisfy set of constraints imposed from universal pole structure:

$$\frac{1}{\epsilon}\log\left(\frac{-m^2}{s}\right) \qquad \frac{1}{\epsilon}\log\left(1-\frac{t}{m^2}\right) \qquad \frac{1}{\epsilon}\log\left(\frac{\beta+1}{\beta-1}\right)$$

- Three IR consistency equations:

$$\begin{aligned}
C^{[L]}_{1|2|3|4} + \langle 2|1|2] C^{[L]}_{3;23|4|1} &= s_{23}\langle 2|1|2] A^{(0)}(1_t, 2, 3, 4_{\bar{t}}) \\
C^{[L]}_{1|2|3|4} + s_{23} C^{[L]}_{3;12|3|4} + s_{23} C^{[L]}_{3;1|2|34} &= s_{23}\langle 2|1|2] A^{(0)}(1_t, 2, 3, 4_{\bar{t}}) \\
C^{[R]}_{1|2|3|4} + \langle 2|1|2] C^{[R]}_{3;2|3|41} &= (s_{23} - 2m^2)\langle 2|1|2] A^{(0)}(1_t, 2, 3, 4_{\bar{t}})
\end{aligned}$$

- Eliminate three-mass triangle from “Right-Moving” primitive

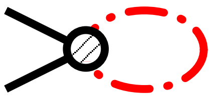
More Integral Functions

- Further changes to the integral basis help to find compact forms and cancel unphysical divergences:
  - $I_2(s,0,m^2) = F_2(s,0,m^2) + I_2(m^2,0,m^2)$
  - $I_2(s,m^2,m^2) = F_2(s,m^2,m^2) + I_2(m^2,0,m^2) - 2$
  - $I_2(s,0,0) = I_2(s,0,0;m^2) + I_2(m^2,0,m^2) - 2$
  - $I_1(m^2) = m^2(I_2(m^2,0,m^2) + 1)$
- This helps to collect all $\log(m^2)$ in one place...
- Cancellation spurious poles with $6-2\epsilon$ box:

$$I_4^{4-2\epsilon} = c_0 I_4^{6-2\epsilon} + \sum_{k=1}^{4} c_k I_3^{(k),4-2\epsilon}$$

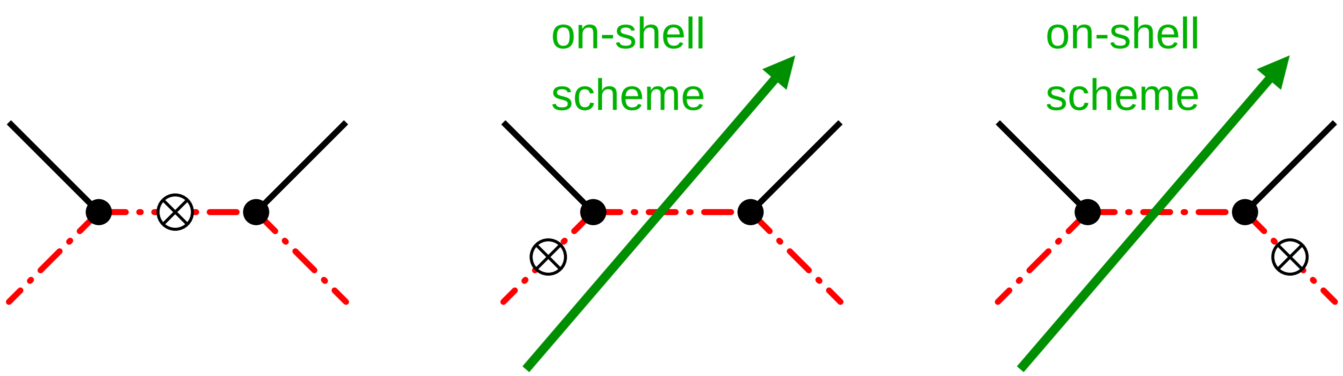
Mass renormalisation

- Amplitude is not gauge invariant before mass renormalisation
- On-shell wave-function cuts are divergant, $I_2(m^2, 0, m^2)$

$c_{2;m^2}$  $+\ c_1$ 

- We used the Feynman computation then changed basis
- Coefficient of $\log(m^2)$ constrained by universal poles

$$c_{2;m^2} I_2(m^2, 0, m^2) + c_1 I_1(m^2) = -\frac{1}{2} A^{(0)} I_2(m^2, 0, m^2) + R_{\mathrm{tad}}$$



The Left-Moving Primitive

$$A_4^{[L]}(1^+,2^+,3^+,4^+) =$$

$$- I_4\left(s_{12},s_{23},0,m^2,m^2,0,0,0,m^2,0\right)\frac{\langle\eta_1\eta_4\rangle[32]^2m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle}$$

$$+ F_2\left(s_{12},0,m^2\right)\left(-\frac{2\langle\eta_1\eta_4\rangle[32]m^3 s_{12}}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2}+\frac{\langle\eta_1 K_{12}K_{32}\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2}\right)$$

$$+ \left(I_2\left(m^2,0,m^2\right)-2\right)\frac{\langle\eta_1\eta_4\rangle[32]m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]}$$

$$-\frac{\left(\langle\eta_1 K_{12}K_{32}\eta_4\rangle+\langle\eta_1\eta_4\rangle\langle 2|1|2]\right)[32]m}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]}-\frac{(\langle\eta_1\eta_4\rangle\langle 2|1|2]+\langle\eta_1 2\rangle\langle\eta_4 3\rangle[32])m}{3\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}$$

The Right-Moving Primitive

$$A_4^{[R]}(1^+,2^+,3^+,4^+) =$$

$$+ I_4\left(s_{12},s_{23},m^2,0,0,m^2,0,m^2,m^2,m^2\right)\left(\frac{\langle\eta_1\eta_4\rangle[32]\left(s_{23}-2m^2\right)m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle} - \frac{\langle\eta_1 K_{12}K_{32}\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle}\right)$$

$$+ I_4^{6-2\epsilon}\left(s_{12},s_{23},m^2,0,0,m^2,0,m^2,m^2,m^2\right)\left(-\frac{\langle\eta_1\eta_4\rangle[23](2m^2+\langle 2|1|2])m^3}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]} - \frac{\langle\eta_1\eta_4\rangle[32]m^3}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle}\right.$$

$$+\frac{2\langle\eta_1 K_{12}K_{32}\eta_4\rangle[32]m^5}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2} - \frac{\langle\eta_1 2\rangle\langle\eta_4 3\rangle[32]^2m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]} + \frac{(\langle\eta_1 K_{12}K_{32}\eta_4\rangle+\langle\eta_1 2\rangle\langle\eta_4 3\rangle[32])[32]m^3}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]}$$

$$\left.+\frac{2\langle\eta_1 2\rangle\langle\eta_4 3\rangle[32]m^3}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}\right) + I_3\left(0,0,s_{23},m^2,m^2,m^2\right)\left(\frac{2(2\langle\eta_1 2\rangle\langle\eta_4 3\rangle-\langle\eta_1\eta_4\rangle\langle 23\rangle)[32]m^5}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|2]}\right.$$

$$\left.+\frac{2\langle\eta_1 K_{12}K_{32}\eta_4\rangle[32]m^5}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2} - \frac{\langle\eta_1\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle}\right) + I_3\left(s_{23},m^2,m^2,m^2,m^2,0\right)\frac{\langle\eta_1 K_{12}K_{32}\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]}$$

$$+ I_3\left(s_{12},0,m^2,0,m^2,m^2\right)\left(\frac{(2\langle\eta_1 2\rangle\langle\eta_4 3\rangle-\langle\eta_1\eta_4\rangle)\langle 23\rangle[32]m^3}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} + \frac{\langle\eta_1 K_{12}K_{32}\eta_4\rangle[32]m^3(1-2\beta^2)}{\beta^2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]}\right)$$

$$+ F_2\left(s_{12},0,m^2\right)\left(\frac{2\langle\eta_1\eta_4\rangle[32]m^3 s_{12}}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2} - \frac{\langle\eta_1 K_{12}K_{32}\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2}\right)$$

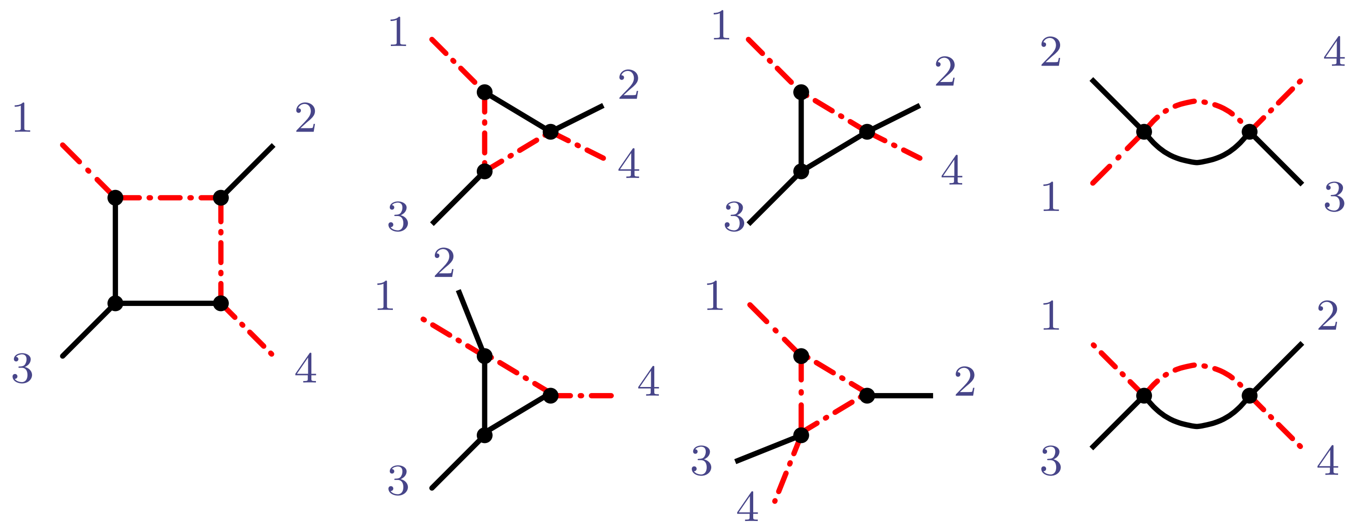
$$-\left(\left(I_2\left(m^2,0,m^2\right)-2\right)\frac{\langle\eta_1\eta_4\rangle[32]m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]} - \frac{\left(\langle\eta_1\eta_4\rangle\langle 2|1|2]+\langle\eta_1 K_{12}K_{32}\eta_4\rangle\right)[32]m}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]}\right.$$

Structure in the Sub-Leading Colour

- $A_4^{[L]}(1_t, 2, 4_{\bar{t}}, 3)$ has full integral structure (bubble, tadpoles, rational terms)
- Only boxes and triangles remain in $A_{4;3}$ after permutation sum is performed.

$$A_{4;3}(1_t, 2, 3, 4_{\bar{t}}) = A_4^{[L]}(1_t, 2, 3, 4_{\bar{t}}) + A_4^{[R]}(1_t, 2, 3, 4_{\bar{t}}) + A_4^{[L]}(1_t, 2, 4_{\bar{t}}, 3) + (2 \leftrightarrow 3)$$

$$A^{[L]}(1_t, 2, 4_{\bar{t}}, 3)$$



The sub-leading colour primitive

$$A_4^{[L]}(1^+,2^+,4^+,3^+) =$$

$$+ I_4\left(s_{12},s_{13},m^2,0,m^2,0,0,m^2,m^2,0\right)\left(\frac{\langle\eta_1 3\rangle\langle\eta_4 3\rangle[32]^2 m^5}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 3|1|2]} - \frac{\langle\eta_1\eta_4\rangle\langle 3|1|3][32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle}\right.$$

$$\left. + \frac{\left(2\langle\eta_1\eta_4\rangle\langle 2|1|2]^2 + \langle\eta_1 3\rangle\langle\eta_4 3\rangle\langle 2|1|3][32]\right)m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}\right)$$

$$+ I_3\left(s_{12},0,m^2,m^2,0,0\right)\left(\frac{m^3\langle\eta_1\eta_4\rangle\langle 2|1|2]}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} - \frac{m^3\langle\eta_1 3\rangle\langle\eta_4 3\rangle\langle 2|1|2][32]}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 3|1|2]}\right)$$

$$+ I_3\left(s_{12},0,m^2,0,m^2,m^2\right)\left(\frac{\langle\eta_1 2\rangle\langle\eta_4 2\rangle\langle 3|1|2][32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|2]} + \frac{\langle\eta_1\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle}\right.$$

$$\left. + \frac{\langle\eta_1 3\rangle\langle\eta_4 3\rangle\langle 2|1|2][32]m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 3|1|2]} + \frac{(\langle\eta_1\eta_4\rangle\langle 2|1|2] - 2\langle\eta_1 2\rangle\langle\eta_4 3\rangle[32])m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}\right)$$

$$+ I_3\left(s_{13},0,m^2,0,m^2,m^2\right)\left(-\frac{2\langle\eta_1\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle} + \frac{\langle\eta_1 3\rangle\langle\eta_4 3\rangle\langle 3|1|3][32]m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 3|1|2]}\right.$$

$$\left. + \frac{\langle\eta_1 2\rangle\langle\eta_4 2\rangle\langle 3|1|2][32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 3|1|3]} - \frac{(\langle\eta_1\eta_4\rangle\langle 2|1|2] - 2\langle\eta_1 2\rangle\langle\eta_4 3\rangle[32])m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}\right)$$

$$+ I_3\left(s_{13},0,m^2,m^2,0,0\right)\left(-\frac{\langle\eta_1 3\rangle\langle\eta_4 3\rangle[32]\langle 3|1|3]m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 3|1|2]} + \frac{\langle\eta_1\eta_4\rangle\langle 3|1|3]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}\right)$$

D-dimensional cuts from massive cuts

- Rational terms from the large mass limit of 4-D cuts [SB (2008)]
- Example: Heavy fermion loop in $gg \to p\bar{p}$ (no tadpole)

$$A^{[H]}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) = \frac{2m\left(\langle\eta_1\eta_4\rangle\langle 2|1|2] + \langle 2\eta_1\rangle\langle 3\eta_4\rangle[32]\right)}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2 s_{23}}$$
$$\left(s_{23} m_H^2 I_3(0,0,s_{23},m_H^2,m_H^2,m_H^2) + 2m_H^2 F_2(s_{23},m_H^2,m_H^2) + R^{[H]}\right)$$

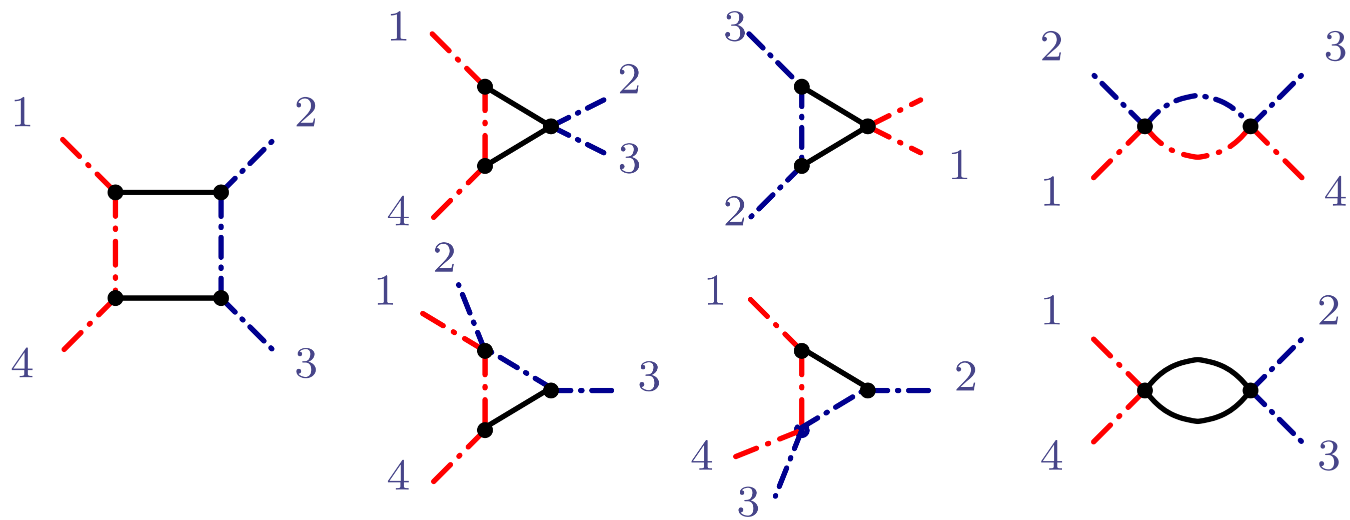
- $m_H^2 \to m_H^2 + \mu^2$, $I_3[\mu^2] = \frac{1}{2}$, $I_2[\mu^2](s) = -\frac{s+3\left(m_1^2+m_2^2\right)}{6}$

$$\Rightarrow R^{[H]} = \frac{s_{23}}{2} - \frac{s_{23} - 6m^2}{6} + \frac{6m^2}{6} = \frac{s_{23}}{6}$$

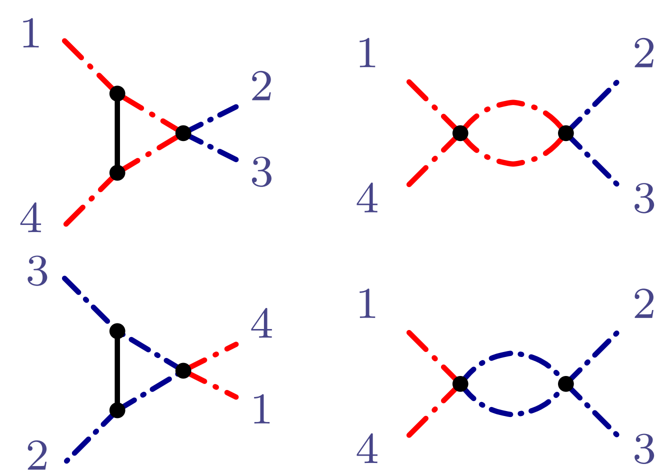
- $\log(m^2)$ coefficient constrained by IR/UV poles [Bern,Morgan]

$q\bar{q} \to t\bar{t}$

$$A^{[lc]}(1_t, 2_q, 3_{\bar{q}}, 4_{\bar{t}})$$



$$A^{[slc]}(1_t, 2_q, 3_{\bar{q}}, 4_{\bar{t}})$$



Remaining amplitudes for $p\bar{p} \to t\bar{t}$

- $gg \to t\bar{t}$: Single additional configuration: $++-+$
- $q\bar{q} \to t\bar{t}$: Single additional configuration: $++-+$
- Additional complexity : ~ 4 times longer expressions

$tgg\bar{t}$	$A^{[L]}$	$A^{[R]}$	$A^{[L,sl]}$	$A^{[H]}$
$++++$	6	25	20	4
$++-+$	80	100	100	0
$tq\bar{q}\bar{t}$	$A^{[lc]}$	$A^{[slc]}$	$A^{[H]}$	-
$++-+$	70	25	4	-

- Cross checks (c++ spinors +`qcdloop`): [Ellis,Zanderighi]
  - Ellis,Giele,Kunszt,Melinkov (D-dimensional cuts,numerical)
  - Körner,Merebashvili (Analytic, Feynman)
  - Anastasiou,Aybat (Analytic, Feynman, $\mathcal{O}(\epsilon^0)$)
- Roughly $\mathcal{O}(10)$ times shorter then KM

Outlook

- On-shell simplifications also persist in massive amplitudes
  - Simplicity in helicity strucutre
  - Cancellations in sub-leading colour
- Hints into improved UV behaviour from QED
- Automated generalised unitarity extraction for arbitary masses
- Compact expressions for $p\bar{p} \to t\bar{t}$ [SB,Sattler,Yundin (in preparation)]
- Ready for future phenomenological applications:
  - Faster evaluation of NLO cross sections
  - Look to include into open source NLO codes:

    [MCFM ; Campbell,Ellis]

    [POWHEG-BOX ; Alioli,Nason,Oleari,Re]
- Higher multiplicity applications in progress

Prospect of NLO Monte-Carlo getting much closer. . .